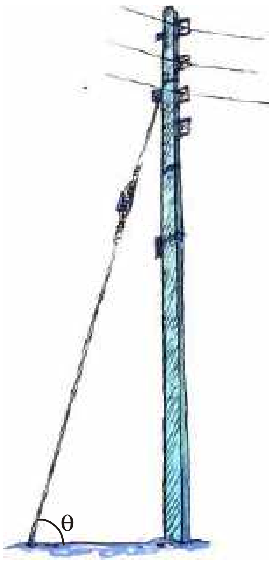
CHAPTER 11 Trigonometry

11.1 INTRODUCTION

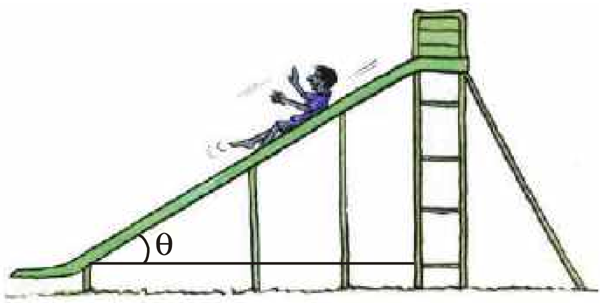
We have seen triangles and their properties in previous classes. There, we observed different daily life situations where we were using triangles.

Let's again look at some of the daily life examples.



- Electric poles are present everywhere. They are usually erected by using a metal wire. The pole, wire and the ground form a triangle. But, if the length of the wire decreases, what will be the shape of the triangle and what will be the angle of the wire with the ground ?
- A person is whitewashing a wall with the help of a ladder which is kept as shown in the adjacent figure on left. If the person wants to paint at a higher position, what will the person do? What will be the change in angle of the ladder with the ground ?

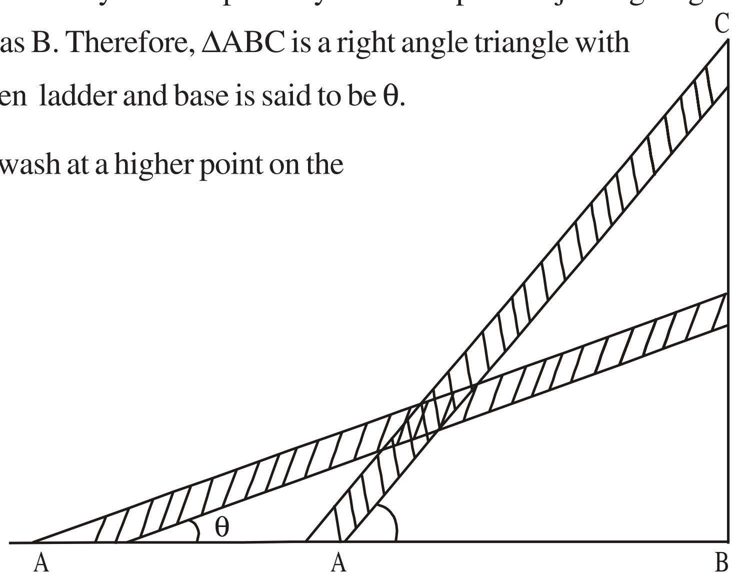
- In the temple at Jainath in Adilabad district, which was built in 13^{th} century, the first rays of the Sun fall at the feet of the Idol of Suryanarayana Swami in the month of December. There is a relation between distance of Idol from the door, height of the hole on the door from which Sun rays are entering and angle of sun rays in that month. Is there any triangle forming in this context?
- In a play ground, children like to slide on slider and slider is on a defined angle from earth. What will happen to the slider if we change the angle? Will children still be able to play on it?



The above examples are geometrically showing the application part of triangles in our daily life and we can measure the heights, distances and slopes by using the properties of triangles. These types of problems are part of 'trigonometry' which is a branch of mathematics.

Now look at the example of a person who is white washing the wall with the help of a ladder as shown in the previous figure. Let us observe the following conditions.

We denote the foot of the ladder by A and top of it by C and the point of joining height of the wall and base of the ladder as B. Therefore, ΔABC is a right angle triangle with right angle at B. The angle between ladder and base is said to be θ.



1. If the person wants to white wash at a higher point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?
2. If the person wants to white wash at a lower point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?

We have observed in the above example of a person who was white washing. When he wants to paint at higher or lower points, he should change the position of ladder. So, when 'θ' is increased, the height also increases and the base decreases. But, when θ is decreased, the height also decreases and the base increases. Do you agree with this statement?

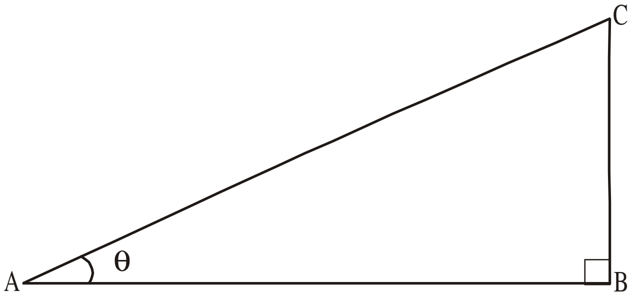
Here, we have seen a right angle triangle ABC and have given ordinary names to all sides and angles. Now let's name the sides again because trigonometric ratios of angles are based on sides only.

11.1.1 Naming The Sides in a Right Triangle

Let's take a right triangle ABC as show in the figure.

In triangle ABC, we can consider $\angle CAB$ as A where angle A is an acute angle. Since AC is the longest side, it is called "hypotenuse".

Here you observe the position of side BC with respect to angle A. It is opposite to angle A and we can call it as "**opposite side of angle A**". And the remaining side AB can be called as "**Adjacent side of angle A**"



AC = Hypotenuse

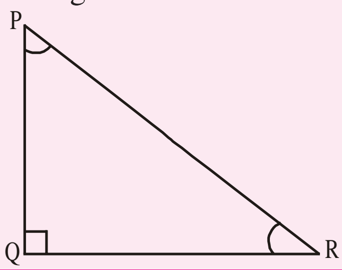
BC = Opposite side of angle A

AB = Adjacent side of angle A

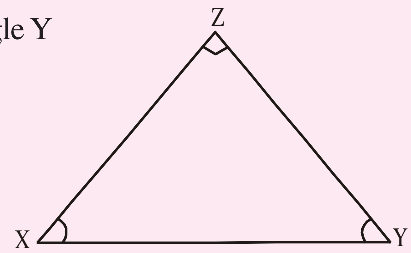
Do This

Identify "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle R



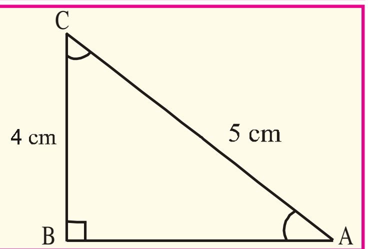
2. (i) For angle X
   (ii) For angle Y



Try This

Write lengths of "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle C
2. For angle A

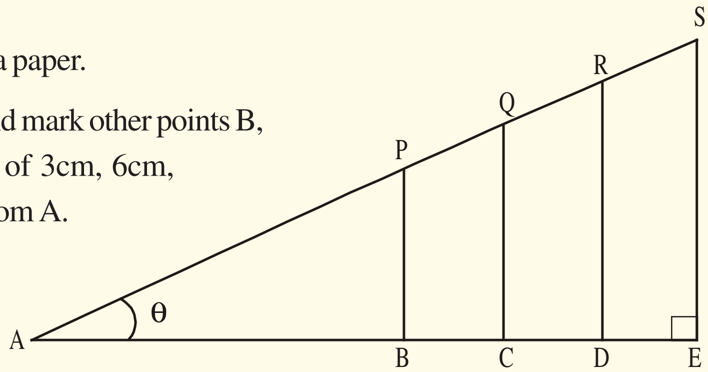


What do you observe? Is there any relation between the opposite side of the angle A and adjacent side of angle C? Like this, suppose you are erecting a pole by giving support of strong ropes. Is there any relationship between the length of the rope and the length of the pole? Here, we have to understand the relationship between the sides and angles we will study this under the section called trigonometric ratios.

11.2 Trigonometric Ratios

We have seen the example problems in the beginning of the chapter which are related to our daily life situations. Let's know about the trigonometric ratios and how they are defined.

Activity



1. Draw a horizontal line on a paper.
2. Let the initial point be A and mark other points B, C, D and E at a distance of 3cm, 6cm, 9cm, 15cm respectively from A.
3. Draw the perpendiculars BP, CQ, DR and ES of lengths 4cm, 8cm, 12cm, 16cm from the points B, C, D and E respectively.
4. Then join AP, PQ, QR and RS.
5. Find lengths of AP, AQ, AR and AS.

Length of hypotenuse	Length of opposite side	Length of adjacent side	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$

Then find the ratios of $\frac{BP}{AP}, \frac{CQ}{AQ}, \frac{DR}{AR}$ and $\frac{ES}{AS}$.

Did you get the same ratio as $\frac{4}{5}$?

Similarly try to find the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$? What do you observe?

11.2.1 Defining Trigonometeric Ratios

In the above activity, when we observe right angle triangles ABP, ACQ, ADR and AES, $\angle A$ is common, $\angle B$, $\angle C$, $\angle D$ and $\angle E$ are right angles and $\angle P$, $\angle Q$, $\angle R$ and $\angle S$ are also equal. Hence, we can say that triangles ABP, ACQ, ADR and AES are similar triangles. When we observe the ratio of opposite side of angle A and hypotenuse in a right angle triangle and the ratio of similar sides in another triangle, it is found to be constant in all the above right angle triangles ABP, ACQ, ADR and AES. And the ratios $\frac{BP}{AP}, \frac{CQ}{AQ}, \frac{DR}{AR}$ and $\frac{ES}{AS}$ can be named as "**sine A**" or simply "**sin A**" in those triangles. If the value of angle A is "x" when it was measured, then the ratio would be "sin x".

Hence, we can conclude that the ratio of opposite side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right angle triangles. This ratio will be named as "sine" of that angle.

Similarly, when we observe the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$, it is also found to be constant. And these are the ratios of the adjacent sides of the angle A and hypotenuses in right angle triangles ABP, ACQ, ADR and AES. So, the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$ will be named as "**cosine A**" or simply "**cos A**" in those triangles. If the value of the angle A is "x", then the ratio would be "cos x"

Hence, we can also conclude that the ratio of the adjacent side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right triangles. This ratio will be named as "cosine" of that angle.

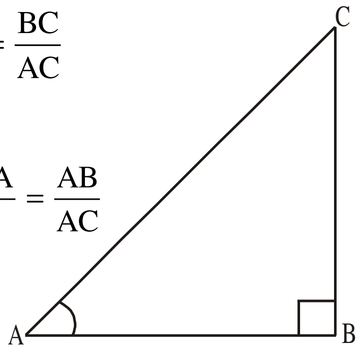
Similarly, the ratio of opposite side and adjacent side of an angle is constant and it can be named as "tangent" of that angle.

Let's Define Ratios in a Right Angle Triangle

Consider a right angle triangle ABC having right angle at B as shown in the following figure.

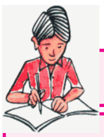
Then, trigonometric ratios of the angle A in right angle triangle ABC are defined as follows :

$$\text{sine of } \angle A = \sin A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of hypotenuse}} = \frac{BC}{AC}$$

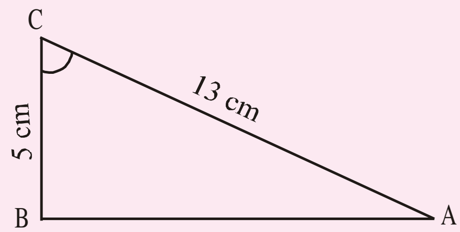


$$\text{cosine of } \angle A = \cos A = \frac{\text{Length of the side adjacent to angle A}}{\text{Length of hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of the side adjacent to angle A}} = \frac{BC}{AB}$$



Do This



1. Find (i) sin C (ii) cos C and (iii) tan C in the adjacent triangle.
2. In a triangle XYZ, $\angle Y$ is right angle, XZ = 17 m and YZ = 15 cm, then find (i) sin X (ii) cos Z (iii) tan X
3. In a triangle PQR with right angle at Q, the value of $\angle P$ is x, PQ = 7 cm and QR = 24 cm, then find $\sin x$ and $\cos x$.

Try This

In a right angle triangle ABC, right angle is at C. $BC + CA = 23$ cm and $BC - CA = 7$cm, then find sin A and tan B.

Think - Discuss

Discuss between your friends that

(i) $\sin x = \frac{4}{3}$ does exist for some value of angle x?

(ii) The value of sin A and cos A is always less than 1. Why?

(iii) tan A is product of tan and A.

There are three more ratios defined in trigonometry which are considered as multiplicative inverse of the above three ratios.

Multiplicative inverse of "sine A" is "cosecant A". Simply written as "cosec A"

i.e., $\text{cosec } A = \dfrac{1}{\sin A}$

Similarly, multiplicative inverses of "cos A" is secant A" (simply written as "sec A") and that of "tan A" is "cotangent A (simply written as cot A)

i.e., $\sec A = \dfrac{1}{\cos A}$ and $\cot A = \dfrac{1}{\tan A}$

How can you define 'cosec' in terms of sides?

If $\sin A = \dfrac{\text{Opposite side of the angle A}}{\text{Hypotenuse}}$,

then $\text{cosec } A = \dfrac{\text{Hypotenuse}}{\text{Opposite side of the angle A}}$

Try This

What will be the ratios of sides for sec A and cot A?

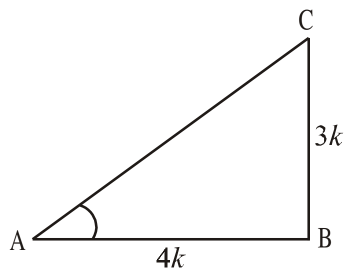
Think - Discuss

- Is $\dfrac{\sin A}{\cos A}$ equal to tan A ?
- Is $\dfrac{\cos A}{\sin A}$ equal to cot A ?

Let us see some examples

Example-1. If $\tan A = \dfrac{3}{4}$, then find the other trigonometric ratio of angle A.

Solution : Given $\tan A = \dfrac{3}{4}$



Hence $\tan A = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{3}{4}$

Therefore, opposite side : adjacent side = 3:4

For angle A, opposite side = BC = $3k$

Adjacent side = AB = $4k$ (where k is any positive number)

Now, we have in triangle ABC (by Pythagoras theorem)

$$AC^2 = AB^2 + BC^2$$
$$= (3k)^2 + (4k)^2 = 25k^2$$
$$AC = \sqrt{25k^2}$$
$$= 5k = \text{Hypotenuse}$$

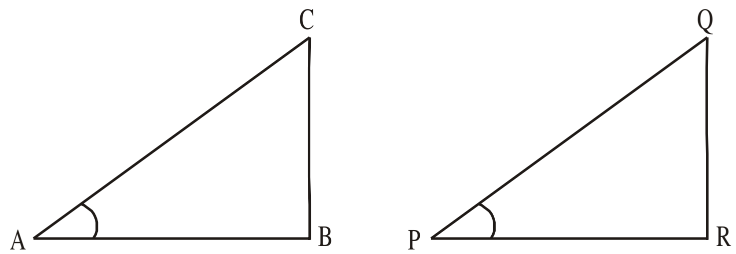
Now, we can easily write the other ratios of trigonometry

$$\sin A = \frac{3k}{5k} = \frac{3}{5} \quad \text{and} \quad \cos A = \frac{4k}{5k} = \frac{4}{5}$$

And also $\text{cosec } A = \frac{1}{\sin A} = \frac{5}{3}$, $\sec A = \frac{1}{\cos A} = \frac{5}{4}$, $\cot A = \frac{1}{\tan A} = \frac{4}{3}$.

Example-2. If $\angle A$ and $\angle P$ are acute angles such that $\sin A = \sin P$ then prove that $\angle A = \angle P$

Solution : Given $\sin A = \sin P$



we have $\sin A = \frac{BC}{AC}$

and $\sin P = \frac{QR}{PQ}$

Then $\frac{BC}{AC} = \frac{QR}{PQ}$

Therefore, $\frac{BC}{AC} = \frac{QR}{PQ} = k$

By using Pythagoras theorem

$$\frac{AB}{PR} = \frac{\sqrt{AC^2 - BC^2}}{\sqrt{PQ^2 - QR^2}} = \frac{\sqrt{AC^2 - k^2BC^2}}{\sqrt{PQ^2 - k^2QR^2}} = \frac{AC}{PQ} \quad \text{(From (1))}$$

Hence, $\frac{AC}{PQ} = \frac{AB}{PR} = \frac{BC}{QR}$ then $\Delta ABC \sim \Delta PQR$

Therefore, $\angle A = \angle P$

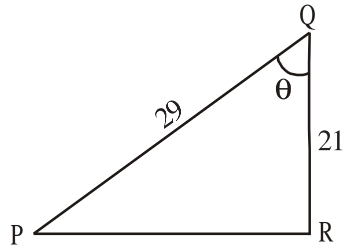
Example-3. Consider a triangle PQR, right angled at P, in which PQ = 29 units, QR = 21 units and $\angle PQR = \theta$, then find the values of

(i) $\cos^2\theta + \sin^2\theta$ and (ii) $\cos^2\theta - \sin^2\theta$

Solution : In PQR, we have

$$PR = \sqrt{PQ^2 - QR^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{400} = 20 \text{ units}$$



$$\sin\theta = \frac{PR}{PQ} = \frac{20}{29}$$

$$\cos\theta = \frac{QR}{PQ} = \frac{21}{29}$$

Now (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{441+400}{841} = 1$

(ii) $\cos^2\theta - \sin^2\theta = \left(\frac{20}{29}\right)^2 - \left(\frac{21}{29}\right)^2 = \frac{-41}{841}$

EXERCISE - 11.1

1. In right angle triangle ABC, 8 cm, 15 cm and 17 cm are the lengths of AB, BC and CA respectively. Then, find out sin A, cos A and tan A.

2. The sides of a right angle triangle PQR are PQ = 7 cm, QR = 25cm and $\angle Q = 90^o$ respectively. Then find, tan Q – tan R.

3. In a right angle triangle ABC with right angle at B, in which $a = 24$ units, $b = 25$ units and $\angle BAC = \theta$. Then, find $\cos\theta$ and $\tan\theta$.

4. If $\cos A = \frac{12}{13}$, then find $\sin A$ and $\tan A$.

5. If $3\tan A = 4$, then find $\sin A$ and $\cos A$.

6. If $\angle A$ and $\angle X$ are acute angles such that cos A = cos X then show that $\angle A = \angle X$.

7. Given $\cot\theta = \frac{7}{8}$, then evaluate (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$ (ii) $\frac{(1+\sin\theta)}{\cos\theta}$

8. In a right angle triangle ABC, right angle is at B, if $\tan A = \sqrt{3}$ then find the value of

   (i) $\sin A \cos C + \cos A \sin C$ (ii) $\cos A \cos C - \sin A \sin C$

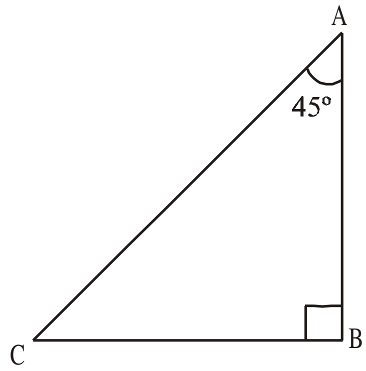
11.3 Trigonometric Ratios of some specific angles

We already know about isosceles right angle triangle and right angle triangle with angles 30º, 60º and 90º.

Can we find sin 30° or tan 60° or cos 45° etc. with the help of these triangles?

Does sin 0° or cos 0° exist?

11.3.1 Trigonometric Ratios of 45º



In isosceles right angle triangle ABC right angled at B

$\angle A = \angle C = 45^{\circ}$ (why ?) and BC = AB (why ?)

Let's assume the length of BC = AB = a

Then, $AC^2 = AB^2 + BC^2$ (by Pythagoras theorem)

$= a^2 + a^2 = 2a^2,$

Therefore, AC = $a\sqrt{2}$

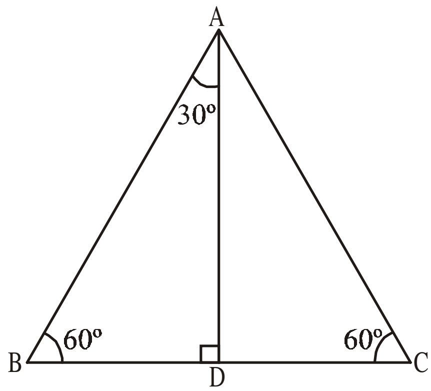
Using the definitions of trigonometric ratios,

$$\sin 45^{\circ} = \frac{\text{Length of the opposite side to angle } 45^{\circ}}{\text{Length of hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45^{\circ} = \frac{\text{Length of the adjacent side to angle } 45^{\circ}}{\text{Length of hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^{\circ} = \frac{\text{Length of the opposite side to angle } 45^{\circ}}{\text{Length of the adjacent side to angle } 45^{\circ}} = \frac{BC}{AC} = \frac{a}{a} = 1$$

Like this you can determine the values of cosec 45°, sec 45° and cot 45°.



11.3.2 Trigonometric Ratios of 30º and 60º

Let us now calculate the trigonometric ratios of 30° and 60°. To calculate them, we will take an equilateral triangle, draw a perpendicular which can divide the triangle into two equal right angle triangles having angles 30°, 60° and 90° in each.

Consider an equilateral triangle ABC. Since each angle is 60^o in an equilateral triangle, we have $\angle A = \angle B = \angle C = 60^o$ and the sides of equilateral triangle is AB = BC = CA = 2a units.

Draw the perpendicular line AD from vertex A to BC as shown in the adjacent figure.

Perpendicular AD acts as "angle bisector of angle A" and "bisector of the side BC" in the equilateral triangle ABC.

Therefore, $\angle BAD = \angle CAD = 30^o$.

Since point D divides the side BC into equal halves,

$BD = \frac{1}{2}BC = \frac{2a}{2} = a$ units.

Consider right angle triangle ABD in the above given figure.

We have AB = 2a and BD = a

Then $AD^2 = AB^2 - BD^2$ by (Pythagoras theorem)

$= (2a)^2 - (a)^2 = 3a^2$.

Therefore, $AD = a\sqrt{3}$

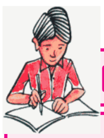
From definitions of trigonometric ratios,

$$\sin 60^o = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

So, similarly $\tan 60^o = \sqrt{3}$ (why?)

Like the above, you can also determine the reciprocals, cosec 60^o, sec 60^o and cot 60^o by using the ratio concepts.



Do This

Find cosec 60^o, sec 60^o and cot 60^o.

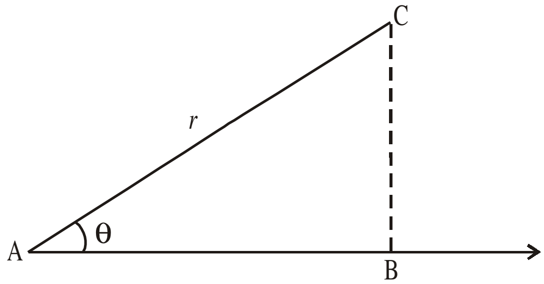
Try This

Find sin 30^o, cos30^o, tan 30^o, cosec 30^o, sec30^o and cot 30^o by using the ratio concepts.

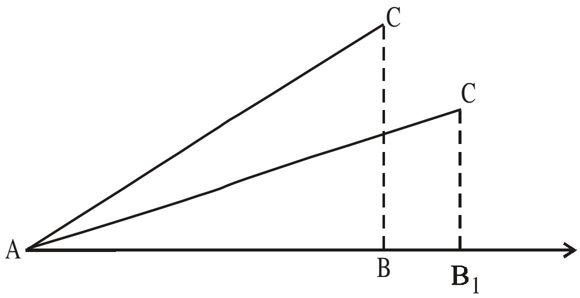
11.3.3 Trigonometric Ratios of 0^o and 90^o

Till now, we have discussed trigonometric ratios of 30^o, 45^o and 60^o. Now let us determine the trigonometric ratios of angles 0^o and 90^o.

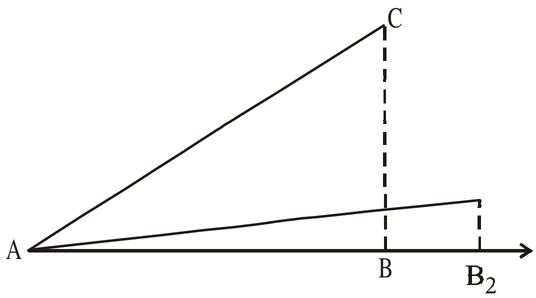
Suppose a segment AC of length r is making an acute angle with ray AB. Height of C from B is BC. When AC leans more on AB so that the angle made by it decreases, then what happens to the lengths of BC and AB ?



As the angle A decreases, the height of C from AB ray decreases and foot B is shifted from B to B_1 and B_2 and gradually when the angle becomes zero, height (i.e. opposite side of the angle) will also become zero (0) and adjacent side would be equal to AC i.e. length equal to *r*.



Step (i)



Step (ii)

Let us look at the trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 0^o$ then $BC = 0$ and $AC = AB = r$

$$\text{then } \sin 0^o = \frac{0}{r} = 0 \text{ and } \cos 0^o = \frac{r}{r} = 1$$

$$\text{we know that } \tan A = \frac{\sin A}{\cos A}$$

$$\text{So, } \tan 0^o = \frac{\sin 0^o}{\cos 0^o} = \frac{0}{1} = 0$$

Think - Discuss

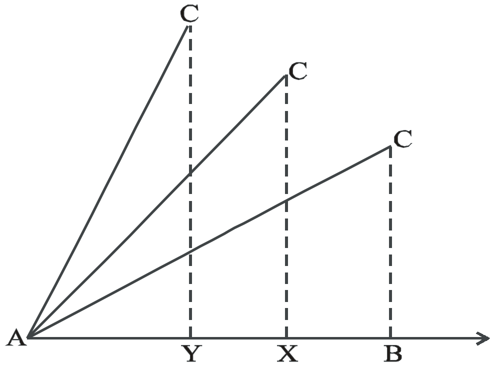
Discuss between your friend about the following conditions:

1. What can you say about $\operatorname{cosec} 0^o = \frac{1}{\sin 0^o}$? Is it defined? Why ?

2. What can you say about $\cot 0^o = \dfrac{1}{\tan 0^o}$. Is it defined? Why ?

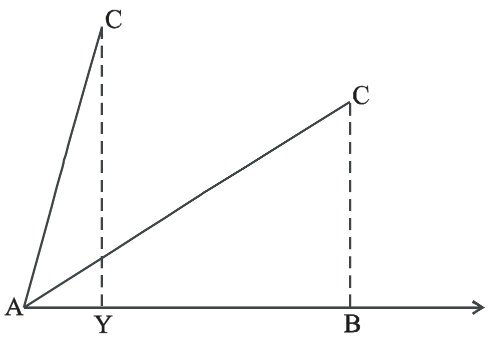
3. $\sec 0^o = 1$. Why ?

Now let us see what happens when angle made by AC with ray AB increases. When angle A is increased, height of point C increases and the foot of the perpendicular shifts from B to X and then to Y and so on. In other words, we can say that the height BC increases gradually, the angle on C gets continuous increment and at one stage the angle reaches 90^o. At that time, point B reaches A and AC equal to BC.

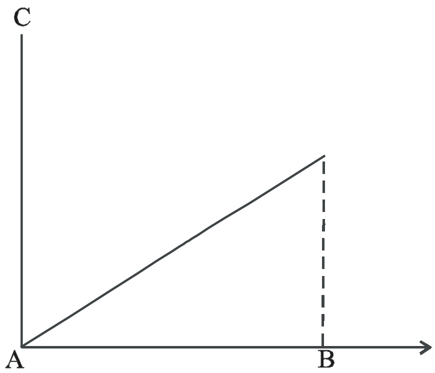


Step (i)

So, when the angle becomes 90^o, base (i.e. adjacent side of the angle) would become zero (0), the height of C from AB ray increases and it would be equal to AC and that is the length equal to r.



Step (ii)



Step (iii)

Now let us see trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 90^o$ then $AB = 0$ and $AC = BC = r$

then $\sin 90^o = \dfrac{r}{r} = 1$ and $\cos 90^o = \dfrac{0}{r} = 0$

TRY THIS

Find the ratios for $\tan 90^o$, $\operatorname{cosec} 90^o$, $\sec 90^o$ and $\cot 90^o$.

Now, let us see the values of trigonometric ratios of all the above discussed angles in the form of a table.

Table 11.1

$\angle A$	0^o	30^o	45^o	60^o	90^o
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

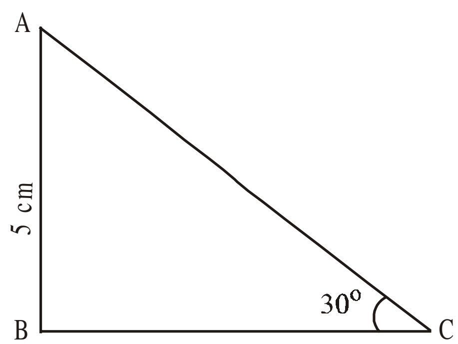
Think - Discuss

What can you say about the values of sin A and cos A, as the value of angle A increases from 0^o to 90^o? (observe the above table)

If $A \geq B$, then $\sin A \geq \sin B$. Is it true ?

If $A \geq B$, then $\cos A \geq \cos B$. Is it true ? Discuss.

Example-4. In ΔABC, right angle is at B, AB = 5 cm and $\angle ACB = 30^o$. Determine the lengths of the sides BC and AC.



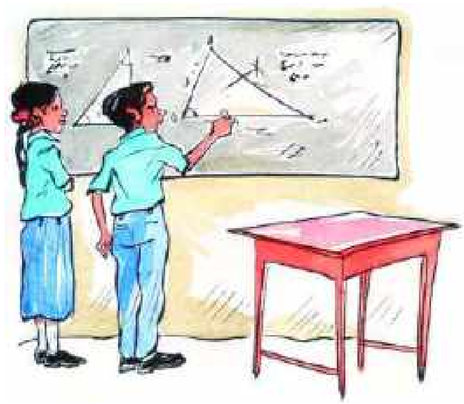
Solution : Given AB=5 cm and $\angle ACB=30^o$. To find the length of side BC, we will choose the trignometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C.

Therefore,

$$\frac{AB}{BC} = \tan C$$

i.e. $\frac{5}{BC} = \tan 30^o = \frac{1}{\sqrt{3}}$

which gives $BC = 5\sqrt{3}$ cm



Now, by using the Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 5^2 + 5\sqrt{3}\,^2$$

$$AC^2 = 25 + 75$$

$$AC = \sqrt{100} \quad = 10 \text{ cm}$$

Example-5. A chord of a circle of radius 6cm is making an angle 60^o at the centre. Find the length of the chord.

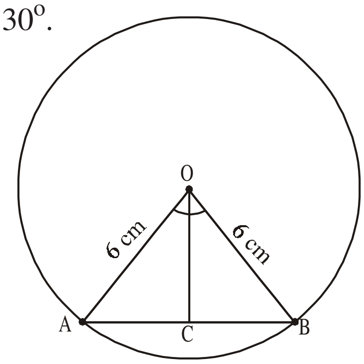
Solution : Given the radius of the circle OA = OB = 6cm

$\angle AOB = 60^o$

OC is height from 'O' upon AB and it is a angle bisector.

then, $\angle COB = 30^o$.

Consider ΔCOB



$\sin 30^o = \frac{BC}{OB}$

$\frac{1}{2} = \frac{BC}{6}$

$BC = \frac{6}{2} = 3$.

But, length of the chord $AB = 2BC$

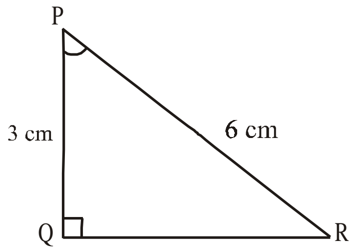
$= 2 \times 3 = 6$ cm

$\therefore$ Therefore, length of the chord = 6 cm

The first use of the idea of **'sine'** in the way we use it today was in the book *Aryabhatiyam* by Aryabhatta, in A.D. 500. Aryabhatta used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation **'*sin*'.**

Example-6. In ΔPQR, right angle is at Q, PQ = 3 cm and PR = 6 cm. Determine $\angle QPR$ and $\angle PRQ$.

Solution : Given PQ = 3 cm and PR = 6 cm



Therefore, $\dfrac{PQ}{PR} = \sin R$

or $\sin R = \dfrac{3}{6} = \dfrac{1}{2}$

So, $\angle PRQ = 30^o$

and therefore, $\angle QPR = 60^o$ (why?)

Note **:** If one of the sides and any other part (either an acute angle or any side) of a right angle triangle is known, the remaining sides and angles of the triangle can be determined.

Example-7. If $\sin(A-B) = \dfrac{1}{2}$, $\cos(A+B) = \dfrac{1}{2}$, $0^o < A+B \le 90^o$, $A > B$, find A and B.

Solution : Since $\sin(A-B) = \dfrac{1}{2}$, therefore, $A-B = 30^o$ (why?)

Also, since $\cos(A+B) = \dfrac{1}{2}$, therefore, $A+B = 60^o$ (why?)

Solving the above equations, we get : $A = 45^o$ and $B = 15^o$. (How?)

Exercise - 11.2

1. Evaluate the following.

(i) $\sin 45^o + \cos 45^o$

(ii) $\dfrac{\cos 45^o}{\sec 30^o + \operatorname{cosec} 60^o}$

(iii) $\dfrac{\sin 30^o + \tan 45^o - \operatorname{cosec} 60^o}{\cot 45^o + \cos 60^o - \sec 30^o}$

(iv) $2\tan^2 45^o + \cos^2 30^o - \sin^2 60^o$

(v) $\dfrac{\sec^2 60^o - \tan^2 60^o}{\sin^2 30^o + \cos^2 30^o}$

2. Choose the right option and justify your choice-

(i) $\dfrac{2\tan 30^o}{1+\tan^2 45^o}$

(a) $\sin 60^o$ (b) $\cos 60^o$ (c) $\tan 30^o$ (d) $\sin 30^o$

(ii) $\dfrac{1-\tan^2 45^o}{1+\tan^2 45^o}$

(a) $\tan 90^o$ (b) 1 (c) $\sin 45^o$ (d) 0

(iii) $\dfrac{2\tan 30^o}{1-\tan^2 30^o}$

(a) $\cos 60^o$ (b) $\sin 60^o$ (c) $\tan 60^o$ (d) $\sin 30^o$

3. Evaluate $\sin 60^o \cos 30^o + \sin 30^o \cos 60^o$. What is the value of $\sin(60^o + 30^o)$. What can you conclude ?

4. Is it right to say $\cos(60^o + 30^o) = \cos 60^o \cos 30^o - \sin 60^o \sin 30^o$.

5. In right angle triangle ΔPQR, right angle is at Q and PQ = 6cms $\angle RPQ = 60^o$. Determine the lengths of QR and PR.

6. In ΔXYZ, right angle is at Y, YZ = x, and XY = $2x$ then determine $\angle YXZ$ and $\angle YZX$.

7. Is it right to say that $\sin(A+B) = \sin A + \sin B$? Justify your answer.

Think - Discuss

For which value of acute angle (i) $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 4$ is true?

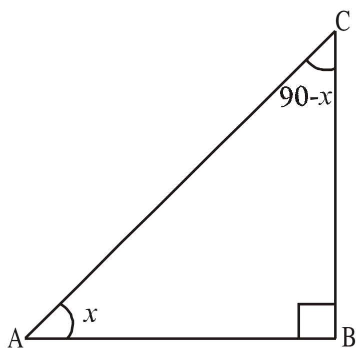
For which value of $0^o \le \theta \le 90^o$, above equation is not defined?

11.4 Trigonometric Ratios of Complementary Angles

We already know that two angles are said to be complementary, if their sum is equal to 90^o. Consider a right angle triangle ABC with right angle at B. Are there any complementary angles in this triangle?

Since angle B is 90^o, sum of other two angles must be 90^o. ($\because$ Sum of angles in a triangle 180^o)

Therefore, $\angle A + \angle C = 90^o$. Hence $\angle A$ and $\angle C$ are said to be complementary angles.



Let us assume that $\angle A = x$, then for angle x, BC is opposite side and AB is adjacent side.

$\sin x = \dfrac{BC}{AC}$ $\cos x = \dfrac{AB}{AC}$ $\tan x = \dfrac{BC}{AB}$

$\text{cosec } x = \dfrac{AC}{BC}$ $\sec x = \dfrac{AC}{AB}$ $\cot x = \dfrac{AB}{BC}$

If $\angle A + \angle C = 90^o$, then we have $\angle C = 90^o - \angle A$

And we have that $\angle A = x$, then $\angle C = 90^o - x$

Let us look at what would be "Opposite side" and "Adjacent side" of the angle $(90^o - x)$in the triangle ABC.

$\sin(90^o - x) = \dfrac{AB}{AC}$ $\cos(90^o - x) = \dfrac{BC}{AC}$ $\tan(90^o - x) = \dfrac{AB}{BC}$

$\text{Cosec}(90^o - x) = \dfrac{AC}{AB}$ $\sec(90^o - x) = \dfrac{AC}{BC}$ $\cot(90^o - x) = \dfrac{BC}{AB}$

Now, if we compare the ratios of angles x and $(90^o - x)$ from the above values of different triginometric terms.

There can be three possibilities in above figure.

$\sin(90^o - x) = \dfrac{AB}{AC} = \cos x$ and $\cos(90^o - x) = \dfrac{BC}{AC} = \sin x$

$\tan(90^o - x) = \dfrac{AB}{BC} = \cot x$ and $\cot(90^o - x) = \dfrac{BC}{AB} = \tan x$

$\text{cosec}(90^o - x) = \dfrac{AC}{AB} = \sec x$ and $\sec(90^o - x) = \dfrac{AC}{BC} = \text{cosec } x$

Think - Discuss

Check and discuss the above relations in the case of angles between 0º and 90º, whether they hold for these angles or not?

So, $\sin(90^o - A) = \cos A$ $\cos(90^o - A) = \sin A$

$\tan(90^o - A) = \cot A$ and $\cot(90^o - A) = \tan A$

$\sec(90^o - A) = \text{cosec } A$ $\text{cosec}(90^o - A) = \sec A$

Now, let us consider some examples

Example-8. Evaluate $\dfrac{\sec 35^\circ}{\operatorname{cosec} 55^\circ}$

Solution : $\operatorname{cosec} A = \sec (90^\circ - A)$

$\operatorname{cosec} 55^\circ = \sec (90^\circ - 35^\circ)$

$\operatorname{cosec} 55^\circ = \sec 35^\circ$

Now $\dfrac{\sec 35^\circ}{\operatorname{cosec} 55^\circ} = \dfrac{\sec 35^\circ}{\sec 35^\circ} = 1$

Example-9. If $\cos 7A = \sin(A - 6^\circ)$, where 7A is an acute angle, find the value of A.

Solution : Given $\cos 7A = \sin(A - 6^\circ)$ ...(1)

$\sin (90 - 7A) = \sin (A - 6^\circ)$

since $(90 - 7A)$ & $(A - 6^\circ)$ are both acute angles,

therefore

$90^\circ - 7A = A - 6^\circ$

$8A = 96^\circ$

which gives $A = 12^\circ$.

Example-10. If $\sin A = \cos B$, then prove that $A + B = 90^\circ$.

Solution : Given that $\sin A = \cos B$ ...(1)

We know $\cos B = \sin (90^\circ - B)$, we can write (1) as

$\sin A = \sin (90^\circ - B)$

If A, B are acute angles, then $A = 90^\circ - B$

$\Rightarrow A + B = 90^\circ$.

Example-11. Express $\sin 81^\circ + \tan 81^\circ$ in terms of trigonometric ratios of angles between 0° and 45°

Solution : We can write $\sin 81^\circ = \cos(90^\circ - 81^\circ) = \cos 9^\circ$

$\tan 81^\circ = \tan(90^\circ - 81^\circ) = \cot 9^\circ$

Then, $\sin 81^\circ + \tan 81^\circ = \cos 9^\circ + \cot 9^\circ$

Example-12. If A, B and C are interior angles of triangle ABC, then show that $\sin\frac{B+C}{2} = \cos\frac{A}{2}$

Solution : Given A, B and C are interior angles of right angle triangle ABC then

$A + B + C = 180^\circ$.

On dividing the above equation by 2 on both sides, we get

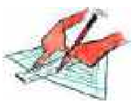
$$\frac{A}{2} + \frac{B+C}{2} = 90^\circ$$

$$\frac{B+C}{2} = 90^\circ - \frac{A}{2}$$

On taking sin ratio on both sides

$$\sin\left(\frac{B+C}{2}\right) = \sin\left(90^\circ - \frac{A}{2}\right)$$

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$$; hence proved.



EXERCISE 11.3

1. Evaluate

(i) $\frac{\tan 36^\circ}{\cot 54^\circ}$ (ii) $\cos 12^\circ - \sin 78^\circ$ (iii) $\text{cosec } 31^\circ - \sec 59^\circ$

(iv) $\sin 15^\circ \sec 75^\circ$ (vi) $\tan 26^\circ \tan 64^\circ$

2. Show that

(i) $\tan 48^\circ \tan 16^\circ \tan 42^\circ \tan 74^\circ = 1$

(ii) $\cos 36^\circ \cos 54^\circ - \sin 36^0 \sin 54^\circ = 0$.

3. If $\tan 2A = \cot(A - 18^\circ)$, where 2A is an acute angle. Find the value of A.

4. If $\tan A = \cot B$ where A and B are acute angles, prove that $A + B = 90^\circ$.

5. If A, B and C are interior angles of a triangle ABC, then show that $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

6. Express $\sin 75^\circ + \cos 65^\circ$ in terms of trigonometric ratios of angles between 0° and 45°.

11.5 Trigonometric identities

We know that an identity is that mathematical equation which is true for all the values of the variables in the equation.

For example $(a+b)^2 = a^2 + b^2 + 2ab$ is an identity.

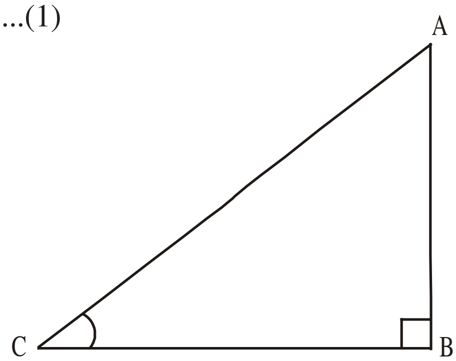
In the same way, an identity equation having trigonometric ratios of an angle is called trigonometric idenity. And it is true for all the values of the angles involved in it.

Here, we will derive a trigonometric identity and remaining would be based on that.

Consider a right angle triangle ABC with right angle is at B, so

From Pythagoras theorem

We have $AB^2 + BC^2 = AC^2$ ....(1)



Dividing each term by AC^2, we get

$$\Rightarrow \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $\left[\frac{AB}{AC}\right]^2 + \left[\frac{BC}{AC}\right]^2 = \left[\frac{AC}{AC}\right]^2$

i.e., $(\cos A)^2 + (\sin A)^2 = 1$

Here, we generally write $\cos^2 A$ in the place of $(\cos A)^2$

i.e., $(\cos A)^2 = \cos^2 A$ (Do not write $\cos A^2$)

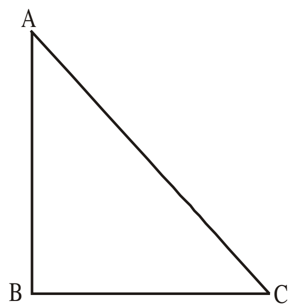
$\therefore$ above equation is $\cos^2 A + \sin^2 A = 1$

We have given an equation having a variable A(angle) and above equation is true for all the value of A. Hence the above equation is a trigonometric identity.

Therefore, we have trigonometric idenity

$$\mathbf{\cos^2 A + \sin^2 A = 1}$$

Let us look at another trigonometric idenity



From equation (1) we have

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$ (Dividing each term by AB^2)

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $1 + \tan^2 A = \sec^2 A$

Similarly, on dividing (1) by BC^2, we get $\cot^2 A + 1 = \text{cosec}^2 A$.

By using above identities, we can express each trigonometric ratio in terms of another ratio. If we know the value of a ratio, we can find all other ratios by using these identities.

Think - Discuss

Are these identities true for $0^0 \le A \le 90^0$? If not, for which values of A they are true?

- $\sec^2 A - \tan^2 A = 1$
- $\text{cosec}^2 A - \cot^2 A = 1$

Do This

(i) If $\sin C = \frac{15}{17}$, then find cos A. (ii) If $\tan x = \frac{5}{12}$, then find sec x.

(iii) If $\text{cosec}\,\theta = \frac{25}{7}$, then find cot x.

Try This

Evaluate the following and justify your answer.

(i) $\dfrac{\sin^2 15^o + \sin^2 75^o}{\cos^2 36^o + \cos^2 54^o}$ (ii) $\sin 5^o \cos 85^o + \cos 5^o \sin 85^o$

(iii) $\sec 16^o \ \text{cosec}\, 74^o - \cot 74^o \tan 16^o$.

Example-13. Show that $\cot\theta + \tan\theta = \sec\theta \ \text{cosec}\,\theta$.

Solution : LHS $= \cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \qquad \text{(why ?)}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\ \cos\theta}$$

$$= \frac{1}{\sin\theta \ \cos\theta} \qquad \text{(why ?)}$$

$$= \frac{1}{\sin\theta} \ \frac{1}{\cos\theta} = \operatorname{cosec}\theta \ \sec\theta$$

Example-14. Show that $\tan^2\theta + \tan^4\theta = \sec^4\theta - \sec^2\theta$

Solution : L.H.S. $= \tan^2\theta + \tan^4\theta$

$= \tan^2\theta\,(1 + \tan^2\theta)$

$= \tan^2\theta \,.\, \sec^2\theta$ (Why ?)

$= (\sec^2\theta - 1)\sec^2\theta$ (Why ?)

$= \sec^4\theta - \sec^2\theta = \text{R.H.S}$

Example-15. Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

Solution : LHS $= \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$ (multiply and divide by $1 + \cos\theta$)

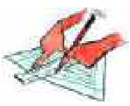
$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}} \qquad \text{(Why ?)}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta = \text{R.H.S.}$$



Exercise 11.4

1. Evaluate the following :

   (i) $(1 + \tan\theta + \sec\theta)\,(1 + \cot\theta - \operatorname{cosec}\theta)$

   (ii) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$

   (iii) $(\sec^2\theta - 1)\,(\operatorname{cosec}^2\theta - 1)$

2. Show that $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

3. Show that $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

4. Show that $\dfrac{1-\tan^2 A}{\cot^2 A - 1} = \tan^2 A$

5. Show that $\dfrac{1}{\cos\theta} - \cos\theta = \tan\theta.\sin\theta$

6. Simplify $\sec A\,(1 - \sin A)\,(\sec A + \tan A)$

7. Prove that $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

8. Simplify $(1 - \cos\theta)\,(1 + \cos\theta)\,(1 + \cot^2\theta)$

9. If $\sec\theta + \tan\theta = p$, then what is the value of $\sec\theta - \tan\theta$?

10. If $\text{cosec}\,\theta + \cot\theta = k$ then prove that $\cos\theta = \dfrac{k^2-1}{k^2+1}$

Optional Exercise

[This exercise is not meant for examination]

1. Prove that $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\text{cosec}\,\theta - 1}{\text{cosec}\,\theta + 1}$

2. Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$ using the identity $\sec^2\theta = 1 + \tan^2\theta$.

3. Prove that $(\text{cosec}\,A - \sin A)\,(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

4. Prove that $\dfrac{1+\sec A}{\sec A} = \dfrac{\sin^2 A}{1-\cos A}$.

5. Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1+\tan A}{1-\cot A}\right)^2 = \tan^2 A$

6. Prove that $\left(\dfrac{(\sec A - 1)}{(\sec A + 1)}\right) = \left(\dfrac{(1-\cos A)}{(1+\cos A)}\right)$

WHAT WE HAVE DISCUSSED

1. In a right angle triangle ABC, right angle is at B,

$$\sin A = \frac{\text{Side opposite to angle A}}{\text{Hypotenuse}}, \quad \cos A = \frac{\text{Side adjacent to angle A}}{\text{Hypotenuse}}$$

2. $\operatorname{cosec} A = \dfrac{1}{\sin A}$; $\sec A = \dfrac{1}{\cos A}$; $\tan A = \dfrac{\sin A}{\cos A}$; $\tan A = \dfrac{1}{\cot A}$

3. If one of the trignometric ratios of an acute angle is known, the remaining trignometric ratios of the angle can be determined.

4. The trignometric ratios for angle 0^o, 30^o, 45^o, 60^o and 90^o.

5. The value of sin A or cos A never exceeds 1, whereas the value of sec A or cosec A is always greater than or equal to 1.

6. $\sin(90^o - A) = \cos A$, $\cos(90^o - A) = \sin A$

   $\tan(90^o - A) = \cot A$, $\cot(90^o - A) = \tan A$

   $\sec A(90^o - A) = \operatorname{cosec} A$, $\operatorname{cosec}(90^o - A) = \sec A$

7. $\sin^2 A + \cos^2 A = 1$

   $\sec^2 A - \tan^2 A = 1$ for $0^o \le A \le 90^o$

   $\operatorname{cosec}^2 A - \cot^2 A = 1$ for $(0^o \le A \le 90^o)$