CHAPTER 3 Polynomials

3.1 INTRODUCTION

Let us observe two situations

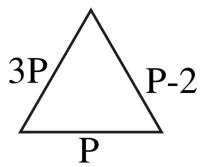
1. A flower bed in a garden is in the shape of a triangle. The longest side is 3 times the middle side and smallest side is 2 units shorter than the middle side. Let **P** represent the length of the middle side, then what's the perimeter in terms of **P?**

2. The length of a rectangular dining hall is twice its breadth. Let x represent the breadth of the hall. What is the area of the floor of the hall in terms of x?

In the above situations, there is an unknown in each statement. In the first situation, middle side is given as '**P**' units.

Since, Perimeter of triangle = sum of all sides

$$\begin{aligned}\text{Perimeter} &= P + 3P + P - 2\\ &= 5P - 2\end{aligned}$$

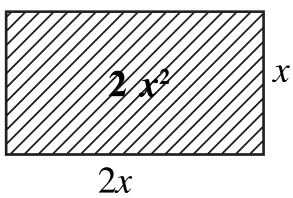


Similarly in the second situation, length is given as twice the breadth.

So, if breadth = x, length = $2x$

Since area of rectangle = lb

$$\begin{aligned}\text{Area} &= (2x)\,(x)\\ &= 2\,x^2\end{aligned}$$



As you know, the perimeter, $5P - 2$ of the triangle and area $2x^2$ of the rectangle are in the form of polynomials of different degrees.

3.2 What are Polynomials?

Polynomials are algebraic expressions constructed using constants and variables. Coefficients operate on variables, which can be raised to various powers of non-negative integer exponents. For example, $2x + 5$, $3x^2 + 5x + 6$, $-5y$, x^3 are some polynomials.

$\frac{1}{x^2}, \frac{1}{\sqrt{2x}}, \frac{1}{y-1}, \sqrt{3x^3}$ etc. are not polynomials.

Why is $\frac{1}{y-1}$ not a polynomial? Discuss with your friends and teacher.

Do This

State which of the following are polynomials and which are not? Give reasons.

(i) $2x^3$ (ii) $\frac{1}{x-1}$ (iii) $4z^2 + \frac{1}{7}$ (iv) $m^2 - \sqrt{2}m + 2$ (v) $P^{-2} + 1$

3.2.1 Degree of a Polynomial

Recall that if p(x) is a polynomial in x, the highest power of x in p(x) is called the degree of the polynomial p(x). For example, $3x + 5$ is a polynomial in the variable x. It is of degree 1 and is called a linear polynomial. $5x$, $\sqrt{2}y+5$, $\frac{1}{3}$P, $m + 1$ etc. are some more linear polynomials.

A polynomial of degree 2 is called a quadratic polynomial. For example, $x^2 + 5x + 4$ is a quadratic polynomial in the variable x. $2x^2 + 3x - \frac{1}{2}$, $p^2 - 1$, $3 - z - z^2$, $y^2 - \frac{y}{3} + \sqrt{2}$ are some examples of quadratic polynomials.

The expression $5x^3 - 4x^2 + x - 1$ is a polynomial in the variable x of degree 3, and is called a cubic polynomial. Some more examples of cubic polynomials are $2 - x^3$, p^3, $\ell^3 - \ell^2 - \ell + 5$.

Try This

Write 3 different quadratic, cubic and 2 linear polynomials with different number of terms.

We can write polynomials of any degree. $7u^6 - \frac{3}{2}u^4 + 4u^2 - 8$ is polynomial of degree 6 and $x^{10} - 3x^8 + 4x^5 + 2x^2 - 1$ is a polynomial of degree 10.

We can write a polynomial in a variable x of a degree n where n is any natural number.

Generally, we say

$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n$ is a polynomial of n^{th} degree, where $a_0, a_1, a_2, \ldots\, a_{n-1}, a_n$ are real coefficients and $a_0 \neq 0$

For example, the general form of a first degree polynomial in one variable x is $ax+b$, where a and b are real numbers and $a \neq 0$.

Try This

1. Write a quadratic polynomial and a cubic polynomial in variable x in the general form.
2. Write a general polynomial $q(z)$ of degree n with coefficients that are $b_0 \ldots b_n$. What are the conditions on $b_0 \ldots b_n$?.

3.2.2 Value of a Polynomial

Now consider the polynomial $p(x) = x^2 - 2x - 3$. What is the value of the polynomial at any point? For example, what is the value at $x = 1$? Putting $x = 1$, in the polynomial, we get $p(1) = (1)^2 - 2(1) - 3 = -4$. The value -4, is obtained by replacing x by 1 in the given polynomial $p(x)$. This is the value of $x^2 - 2x - 3$ at $x = 1$.

Similarly, $p(0) = -3$ is the value of $p(x)$ at $x = 0$.

Thus, if $p(x)$ is a polynomial in x, and if k is a real number, then the value obtained by replacing x by k in $p(x)$, is called the value of $p(x)$ at $x = k$, and is denoted by $p(k)$.

Do This

(i) $p(x) = x^2 - 5x - 6$, find the values of $p(1), p(2), p(3), p(0), p(-1), p(-2), p(-3)$.

(ii) $p(m) = m^2 - 3m + 1$, find the value of $p(1)$ and $p(-1)$.

3.2.3 Zeroes of a Polynomial

What are values of $p(x) = x^2 - 2x - 3$ at $x = 3, -1$ and 2?

We have, $p(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$

Also $p(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

and $p(2) = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$

We see that $p(3) = 0$ and $p(-1) = 0$. These points, $x = 3$ and $x = -1$, are called **Zeroes** of the polynomial $p(x) = x^2 - 2x - 3$.

As $p(2) \neq 0$, 2 is not the zero of $p(x)$.

More generally, a real number k is said to be a zero of a polynomial $p(x)$, if $p(k) = 0$.

Do this

(i) Let $p(x) = x^2 - 4x + 3$. Find the value of $p(0)$, $p(1)$, $p(2)$, $p(3)$ and obtain zeroes of the polynomial $p(x)$.

(ii) Check whether -3 and 3 are the zeroes of the polynomial $x^2 - 9$.

Exercise - 3.1

1. (a) If $p(x) = 5x^7 - 6x^5 + 7x - 6$, find

   (i) coefficient of x^5 (ii) degree of $p(x)$ (iii) constant term.

   (b) Write three more polynomials and create three questions for each of them.

2. State which of the following statements are true and which are false? Give reasons for your choice.

   (i) The degree of the polynomial $\sqrt{2}\,x^2 - 3x + 1$ is $\sqrt{2}$.

   (ii) The coefficient of x^2 in the polynomial $p(x) = 3x^3 - 4x^2 + 5x + 7$ is 2.

   (iii) The degree of a constant term is zero.

   (iv) $\dfrac{1}{x^2 - 5x + 6}$ is a quadratic polynomial.

   (v) The degree of a polynomial is one more than the number of terms in it.

3. If $p(t) = t^3 - 1$, find the values of $p(1), p(-1), p(0), p(2), p(-2)$.

4. Check whether -2 and 2 are the zeroes of the polynomial $x^4 - 16$

5. Check whether 3 and -2 are the zeroes of the polynomial $p(x)$ when $p(x) = x^2 - x - 6$.

3.3 Working with Polynomials

You have already studied how to find the zeroes of a linear polynomial.

For example, if k is a zero of $p(x) = 2x + 5$, then $p(k) = 0$ gives $2k + 5 = 0$ i.e., $k = \frac{-5}{2}$.

In general, if k is a zero of $p(x) = ax+b$, $a \neq 0$.

then $p(k) = ak + b = 0$,

i.e., $k = \frac{-b}{a}$, or the zero of the linear polynomial $ax + b$ is $\frac{-b}{a}$.

Thus, the zero of a linear polynomial is related to its coefficients, including the constant term.

Are the zeroes of higher degree polynomials also related to their coefficients? Think about this and discuss with friends. We will come to this later.

3.4 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number k is a zero of the polynomial $p(x)$ if $p(k) = 0$. Let us see the **graphical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

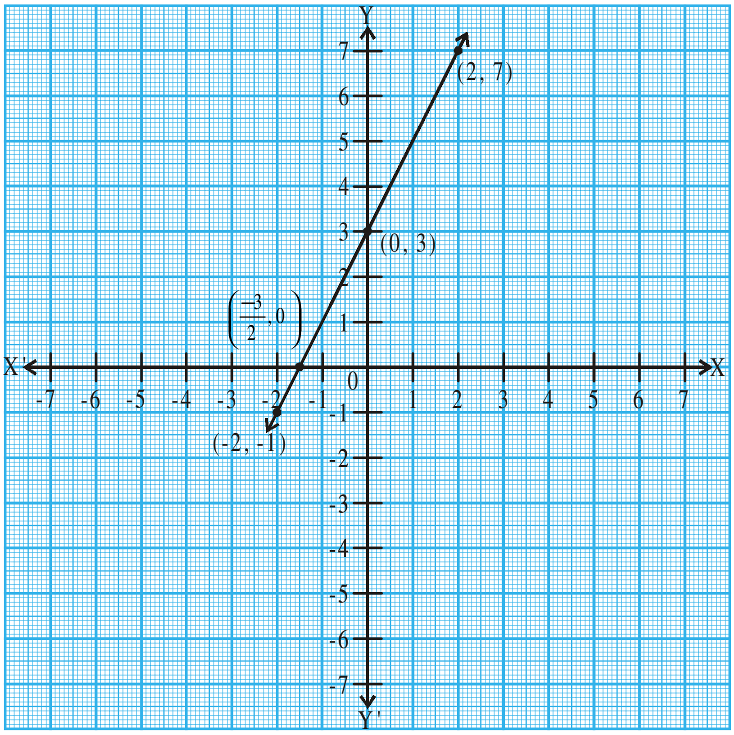
3.4.1. Graphical representation of a linear polynomial

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class-IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line intersecting the y-axis at (0, 3) and passing through the points (–2, –1) and (2, 7).

Table 3.1

x	–2	0	2
$y = 2x + 3$	–1	3	7
(x, y)	(–2, –1)	(0, 3)	(2, 7)

From the graph, you can see that the graph of $y = 2x+3$ intersects the x-axis between $x = -1$ and $x = -2$, that is, at the point $\left(\frac{-3}{2}, 0\right)$. But $x = \frac{-3}{2}$ is also the zero of the polynomial $2x + 3$. Thus, the zero of the polynomial $2x + 3$ is the x-coordinate of the point where the graph of $y = 2x + 3$ intersects the x-axis.



Do This

Draw the graph of (i) $y = 2x + 5$, (ii) $y = 2x - 5$, (iii) $y = 2x$ and find the point of intersection on x-axis. Is the x-coordinates of these points also the zero of the polynomial?

In general, for a linear polynomial $ax + b, a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$.

Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

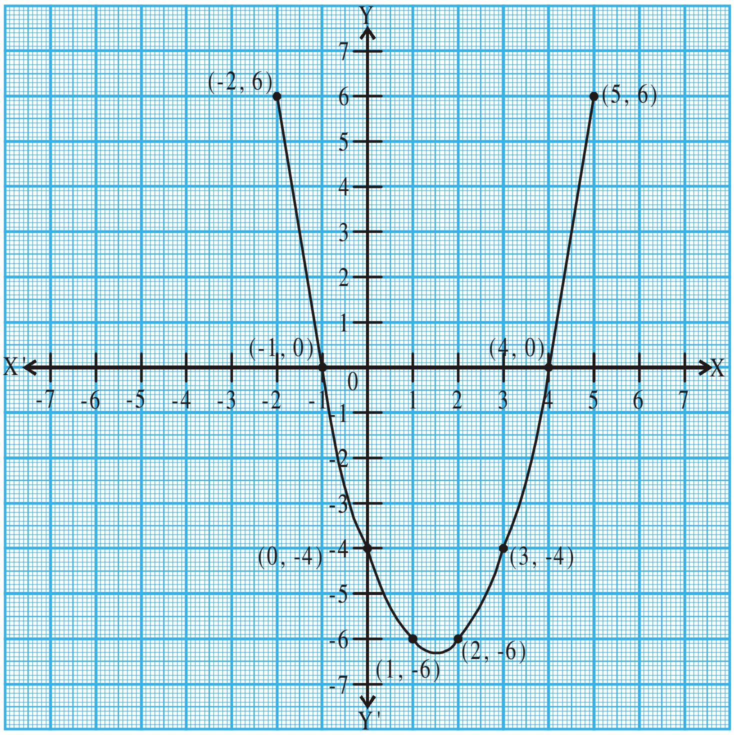
3.4.2. Graphical Representation of a Quadratic Polynomial

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see how the graph of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for x as given in Table 3.2.

Table 3.2

x	-2	-1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	-4	-6	-6	-4	0	6
(x, y)	$(-2, 6)$	$(-1, 0)$	$(0, 4)$	$(1, -6)$	$(2, -6)$	$(3, -4)$	$(4, 0)$	$(5, 6)$

We locate the points listed above on a graph paper and draw the graph.



Is the graph of this quadratic polynomial a straight line? It is like a $\cup$ shaped curve. It intersects the x-axis at two points.

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ either opens upwards like $\cup$ or opens downwards like $\cap$. This depends on whether $a > 0$ or $a < 0$. (The shape of these curves are called **parabolas.**)

We observe that -1 and 4 are zeroes of the quadratic polynomial and -1 and 4 are intersection points of x-axis. Zeroes of the quadratic polynomial $x^2 - 3x - 4$ are the x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis.

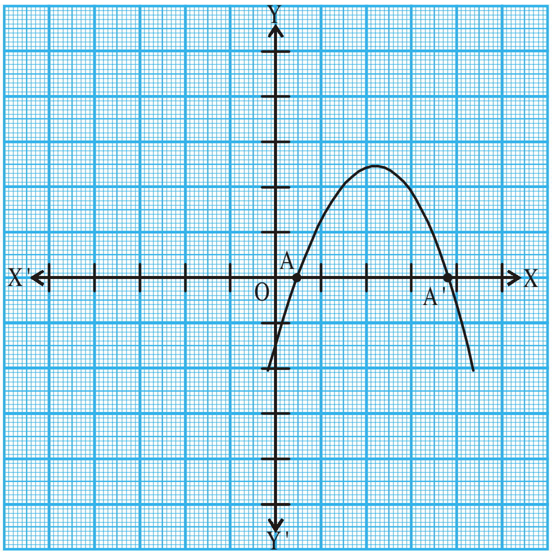
This is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis.

TRY THIS

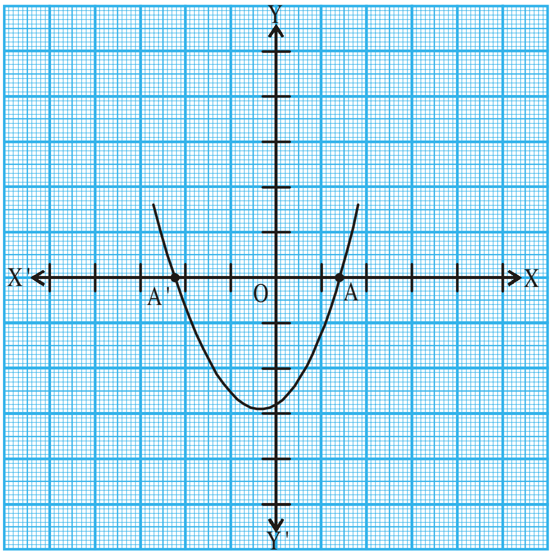
Draw the graphs of (i) $y = x^2 - x - 6$ (ii) $y = 6 - x - x^2$ and find zeroes in each case. What do you notice?

From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

Case (i) : Here, the graph cuts x-axis at two distinct points A and A'. In this case, the x-coordinates of A and A' are the **two zeroes** of the quadratic polynomial ax^2+bx+c . The parabola can open either upward or downward.

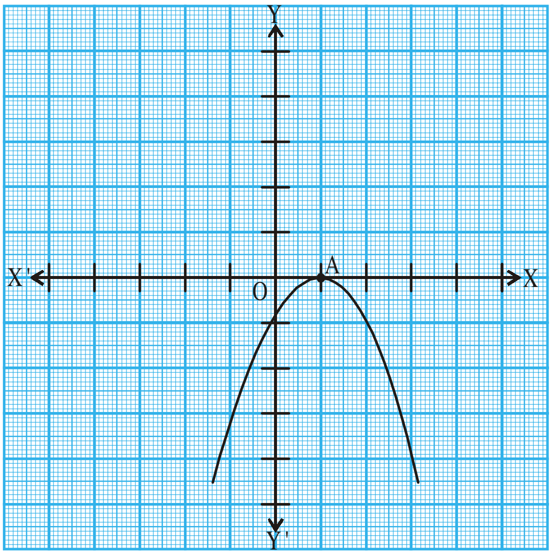


(i)

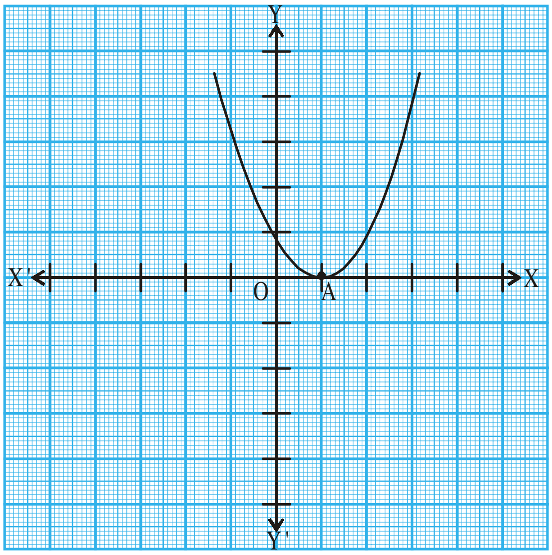


(ii)

Case (ii) : Here, the graph touches x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A' of Case (i) coincide here to become one point A.



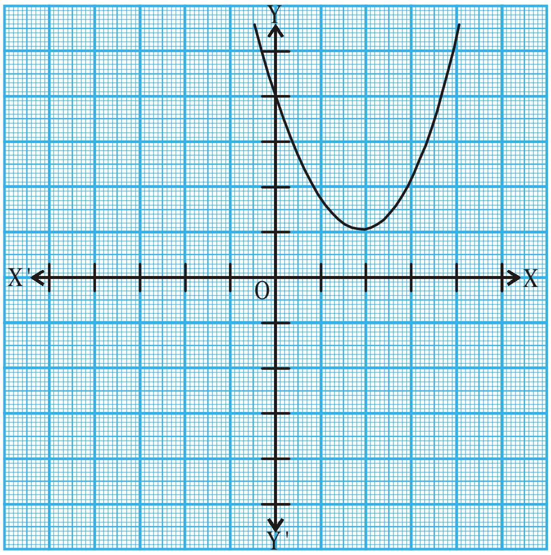
(i)



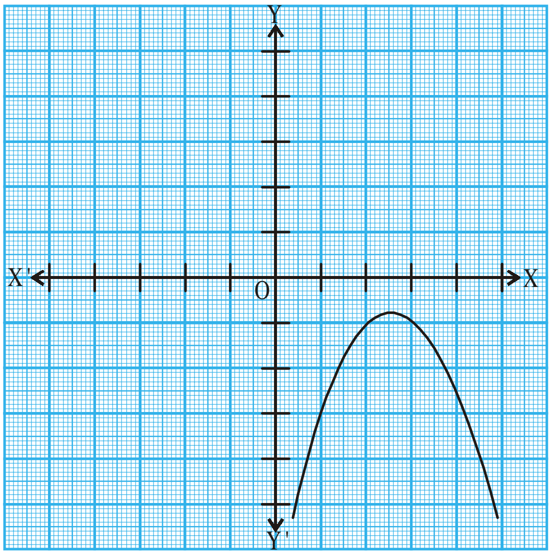
(ii)

The x-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

Case (iii) : Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point.



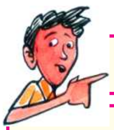
(i)



(ii)

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.



TRY THIS

1. Write three polynomials that have 2 zeros each.
2. Write one polynomial that has one zero.
3. How will you verify if it has only one zero.
4. Write three polynomials that have no zeroes for x that are real numbers.

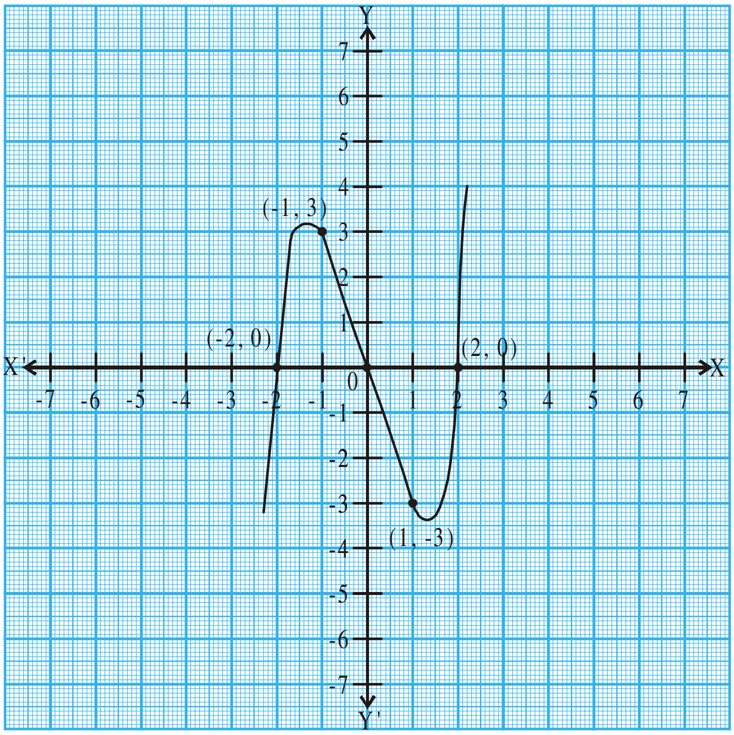
3.4.3 Geometrical Meaning of Zeroes of a Cubic Polynomial

What could you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see how the graph of $y = x^3 - 4x$ looks like, let us list a few values of y corresponding to a few values for x as shown in Table 3.3.

Table 3.3

x	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0
(x, y)	$(-2, 0)$	$(-1, 3)$	$(0, 0)$	$(1, -3)$	$(2, 0)$

On drawing the graph, we see that the graph of $y = x^3 - 4x$ looks like the one given in the figure.



We see from the table above that -2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. -2, 0 and 2 are the x-coordinates of the points where the graph of $y = x^3 - 4x$ intersects the x-axis. So this polynomial has three zeros.

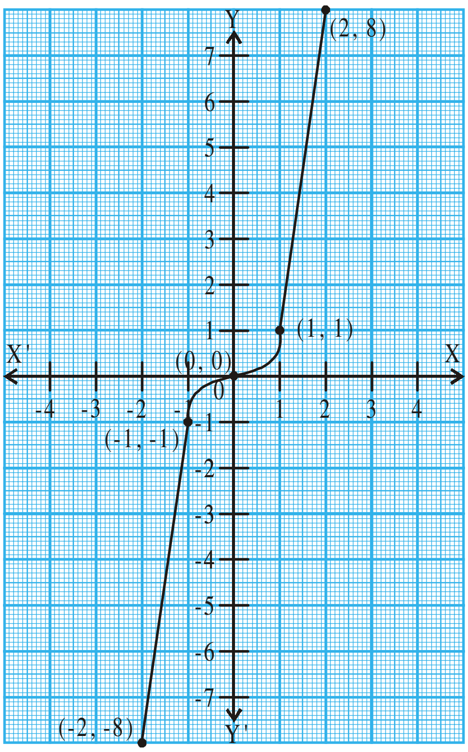
Let us take a few more examples. Consider the cubic polynomials x^3 and $x^3 - x^2$ respectively. See Table 3.4 and 3.5.

Table 3.4

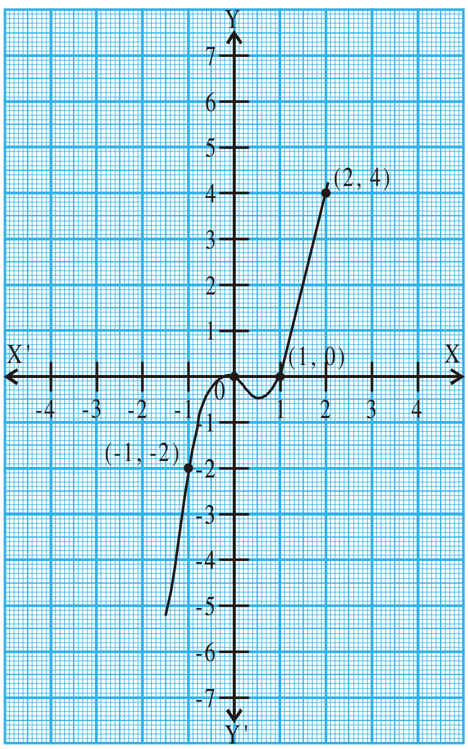
x	-2	-1	0	1	2
$y = x^3$	-8	-1	0	1	8
(x, y)	$(-2, -8)$	$(-1, -1)$	$(0, 0)$	$(1, 1)$	$(2, 8)$

Table 3.5

x	-2	-1	0	1	2
$y = x^3 - x^2$	-12	-2	0	0	4
(x, y)	$(-2, -12)$	$(-1, -2)$	$(0, 0)$	$(1, 0)$	$(2, 4)$



$y = x^3$



$y = x^3 - x^2$

In $y = x^3$, you can see that 0 is the x-coordinate of the only point where the graph of $y = x^3$ intersects the x-axis. So, the polynomial has only one distinct zero. Similarly, 0 and 1 are the x-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the x-axis. So, the cubic polynomial has two distinct zeros.

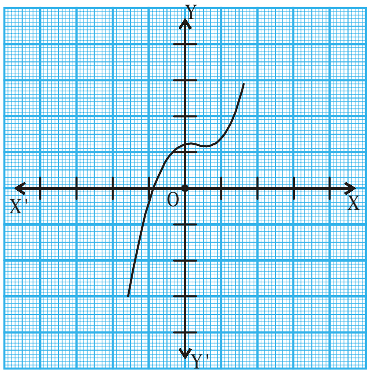
From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

TRY THIS

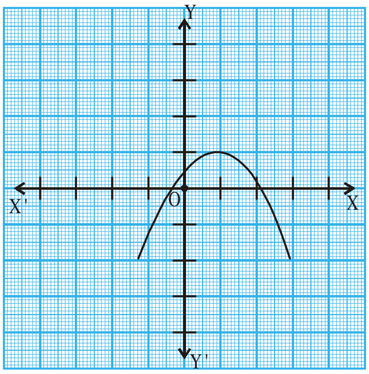
Find the zeroes of cubic polynomials (i) $-x^3$ (ii) $x^2 - x^3$ (iii) $x^3 - 5x^2 + 6x$ without drawing the graph of the polynomial.

Remark : In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at at most n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

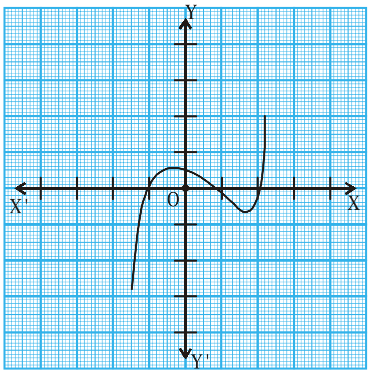
Example-1. Look at the graphs in the figures given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. In each of the graphs, find the number of zeroes of $p(x)$ in the given range of x.



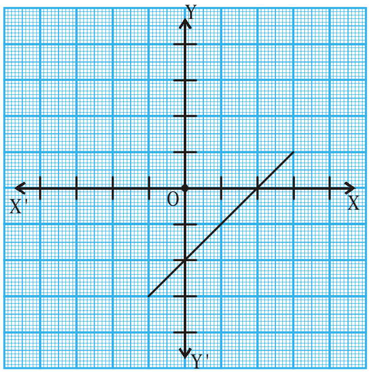
(i)



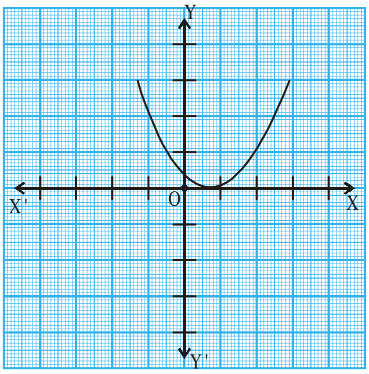
(ii)



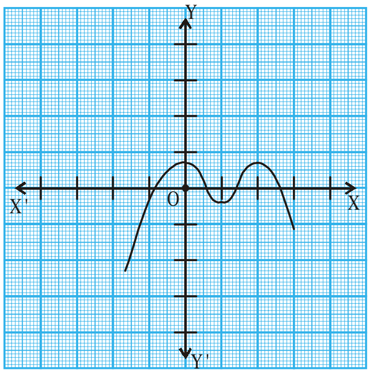
(iii)



(iv)



(v)



(vi)

Solution : In the given range of x in respective graphs :

(i) The number of zeroes is 1 as the graph intersects the x-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the x-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

Example-2. Find the number of zeroes of the given polynomials. And also find their values.

(i) $p(x) = 2x + 1$ (ii) $q(y) = y^2 - 1$ (iii) $r(z) = z^3$

Solution : We will do this without plotting the graph.

(i) $p(x) = 2x + 1$ is a linear polynomial. It has only one zero.

To find zeroes,

Let $p(x) = 0$

So, $2x+1=0$

Therefore $x = \frac{-1}{2}$

The zero of the given polynomial is $\frac{-1}{2}$.

(ii) $q(y) = y^2 - 1$ is a quadratic polynomial.

It has atmost two zeroes.

To find zeroes, let $q(y) = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow (y + 1)(y - 1) = 0$

$\Rightarrow y = -1$ or $y = 1$

Therefore the zeroes of the polynomial are -1 and 1.

(iii) $r(z) = z^3$ is a cubic polynomial. It has at most three zeroes.

Let $r(z) = 0$

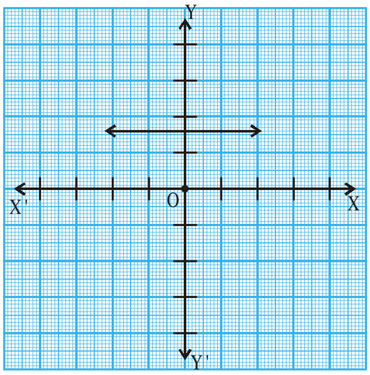
$\Rightarrow z^3 = 0$

$\Rightarrow z = 0$

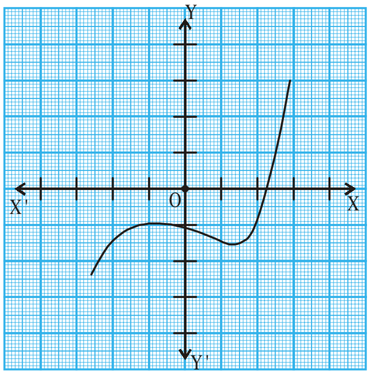
So, the zero of the polynomial is 0.

Exercise – 3.2

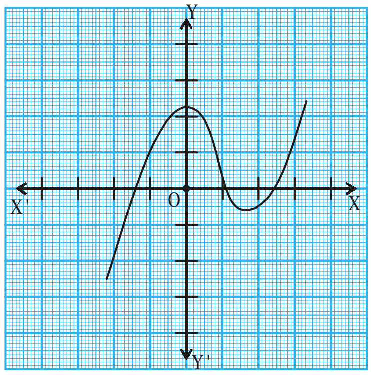
1. The graphs of $y = p(x)$ are given in the figure below, for some polynomials $p(x)$. In each case, find the number of zeroes of $p(x)$.



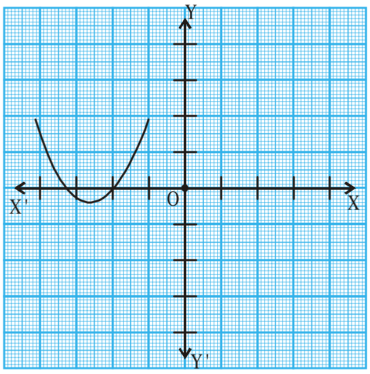
(i)



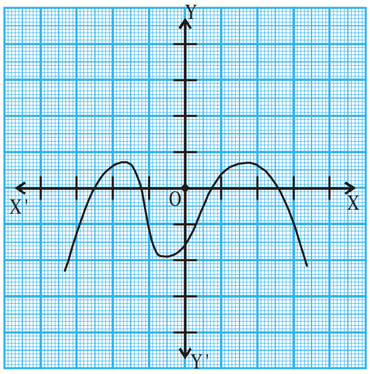
(ii)



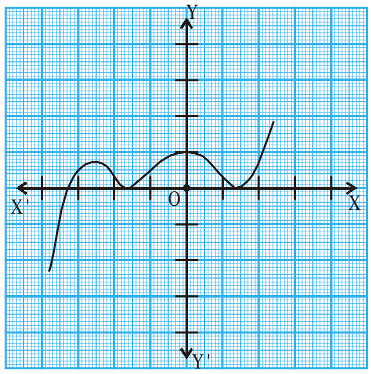
(iii)



(iv)



(v)



(vi)

2. Find the zeroes of the given polynomials.

   (i) $p(x) = 3x$ (ii) $p(x) = x^2 + 5x + 6$

   (iii) $p(x) = (x+2)(x+3)$ (iv) $p(x) = x^4 - 16$

3. Draw the graphs of the given polynomial and find the zeroes. Justify the answers.

   (i) $p(x) = x^2 - x - 12$ (ii) $p(x) = x^2 - 6x + 9$

   (iii) $p(x) = x^2 - 4x + 5$ (iv) $p(x) = x^2 + 3x - 4$

   (v) $p(x) = x^2 - 1$

4. Why are $\frac{1}{4}$ and -1 zeroes of the polynomials $p(x) = 4x^2 + 3x - 1$?

3.5 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\frac{b}{a}$. We will now try to explore the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take the quadratic polynomial $p(x) = 2x^2 - 8x + 6$.

In Class-IX, we have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we split the middle term '$-8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write

$$\begin{aligned} 2x^2 - 8x + 6 &= 2x^2 - 6x - 2x + 6 \\ &= 2x(x-3) - 2(x-3) \\ &= (2x-2)(x-3) = 2(x-1)(x-3) \end{aligned}$$

$p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. We now try and see if these zeroes have some relationship to the coefficients of terms in the polynomial. The coefficient of x^2 is 2; of x is -8 and the constant is 6, which is the coefficient of x^0. (i.e. $6x^0 = 6$)

We see that the sum of the zeroes $= 1 + 3 = 4 = \dfrac{-(-8)}{2} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= 1 \times 3 = 3 = \dfrac{6}{2} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Let us take one more quadratic polynomial:

$$p(x) = 3x^2 + 5x - 2.$$

By splitting the middle term we see,

$$\begin{aligned} 3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 = 3x(x+2) - 1(x+2) \\ &= (3x-1)(x+2) \end{aligned}$$

$3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$

i.e., when $x = \frac{1}{3}$ or $x = -2$.

The zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and -2. We can see that the :

Sum of its zeroes $= \dfrac{1}{3} + (-2) = \dfrac{-5}{3} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of its zeroes $= \dfrac{1}{3} \times (-2) = \dfrac{-2}{3} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Do This

Find the zeroes of the quadratic polynomials given below. Find the sum and product of the zeroes and verify relationship to the coefficients of terms in the polynomial.

(i) $p(x) = x^2 - x - 6$ (ii) $p(x) = x^2 - 4x + 3$

(iii) $p(x) = x^2 - 4$ (iv) $p(x) = x^2 + 2x + 1$

In general, if α and β are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, where $a \neq 0$, then $(x - \alpha)$ and $(x - \beta)$ are the factors of $p(x)$. Therefore,

$$ax^2 + bx + c = k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant}$$

$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

$$= kx^2 - k(\alpha + \beta)x + k\alpha\beta$$

Comparing the coefficients of x^2, x and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha+\beta) \text{ and } c = k\alpha\beta.$$

This gives $\alpha + \beta = \dfrac{-b}{a}$,

$$\alpha\beta = \frac{c}{a}$$

Note **:** α and β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

So, sum of zeroes of a quadratic polynomial $= \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes of a quadratic polynomial $= \alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Let us consider some examples.

Example-3. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

Solution : We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$,

i.e., when $x = -2$ or $x = -5$.

Therefore, the zeroes of $x^2 + 7x + 10$ are -2 and -5.

Now, sum of the zeroes $= -2 + (-5) = -(7) = \dfrac{-(7)}{1} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= -2 \times (-5) = 10 = \dfrac{10}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-4. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution : Recall the identity $a^2 - b^2 = (a - b)(a + b)$.

Using it, we can write:

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Sum of the zeroes $= \sqrt{3} + (-\sqrt{3}) = 0 = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes $= (\sqrt{3}) \times (-\sqrt{3}) = -3 = \dfrac{-3}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-5. Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution : Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and $$\alpha\beta = 2 = \frac{c}{a}.$$

If we take $a = 1$, then $b = 3$ and $c = 2$

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

Similarly, we can take 'a' to be any real number. Let us say it is k. This gives $\frac{-b}{k} = -3$ or b = 3k and $\frac{c}{k} = 2$ or $c = 2k$. Putting the values of a, b and c, we get the polynomial is $kx^2 + 3kx + 2k$.

Example-6. Find a quadratic polynomial if the zeroes of it are 2 and $\frac{-1}{3}$ respectively.

Solution : Let the quadratic polynomial be

$ax^2 + bx + c$, $a \neq 0$ and its zeroes be α and β.

Here $\alpha = 2$, $\beta = \frac{-1}{3}$

Sum of the zeroes $= (\alpha + \beta) = 2 + \left(\frac{-1}{3}\right) = \frac{5}{3}$

Product of the zeroes $= (\alpha\beta) = 2\left(\frac{-1}{3}\right) = \frac{-2}{3}$

Therefore the quadratic polynomial $ax^2 + bx + c$ is

$k[x^2 - (\alpha+\beta)x + \alpha\beta]$, where k is a constant

$$= k[x^2 - \frac{5}{3}x - \frac{2}{3}]$$

We can put different values of k.

When $k = 3$, the quadratic polynomial will be $3x^2 - 5x - 2$.

Try This

(i) Find a quadratic polynomial with zeroes -2 and $\frac{1}{3}$.

(ii) What is the quadratic polynomial whose sum of zeroes is $\frac{-3}{2}$ and the product of zeroes is -1.

3.6 Cubic Polynomials

Let us now look at cubic polynomials. Do you think some relation holds between the zeroes of a cubic polynomial and its coefficients as well?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

We see that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}$.

Since $p(x)$ can have at most three zeroes, these are the zeroes of $2x^3 - 5x^2 - 14x + 8$.

Sum of its zeroes $= 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{coefficient of } x^2)}{\text{coefficient of } x^3}$

Product of its zeroes $= 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-(\text{constant term})}{\text{coefficient of } x^3}$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have:

$$= \{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{constant of } x}{\text{coefficient of } x^3}$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\beta\gamma = \frac{-d}{a}.$$

> $ax^3 + bx^2 + cx + d$ is a polynomial with zeroes α, β, γ. Let us see how α, β, γ relate to a, b, c, d.
>
> Since α, β, γ are the zeroes, the polynomial can be written as $(x-\alpha)(x-\beta)(x-\gamma)$
>
> $$= x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\alpha\gamma) - \alpha\beta\gamma$$
>
> To compare with the polynomial, we multiply by 'a' and get
>
> $$ax^3 - x^2a(\alpha+\beta+\gamma) + xa(\alpha\beta+\beta\gamma+\alpha\gamma) - a\alpha\beta\gamma.$$
>
> $$\therefore\ b = -a(\alpha+\beta+\gamma), c = a(\alpha\beta+\beta\gamma+\alpha\gamma), d = -a\alpha\beta\gamma$$

Do This

If α, β, γ are the zeroes of the given cubic polynomials, find the values as given in the table.

S.No.	Cubic Polynomial	$\alpha+\beta+\gamma$	$\alpha\beta+\beta\alpha+\gamma\alpha$	$\alpha\beta\gamma$
1	x^3+3x^2-x-2			
2	$4x^3+8x^2-6x-2$			
3	x^3+4x^2-5x-2			
4	x^3+5x^2+4			

Let us consider an example.

Example-7. Verify that $3, -1, -\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get

$a = 3,\ b = -5,\ c = -11,\ d = -3$. Further

$$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$$

$$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$$

$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, –1, and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So, we take $\alpha = 3$, $\beta = -1$ and $\gamma = -\frac{1}{3}$.

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

Exercise – 3.3

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

   (i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

   (iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

2. Find the quadratic polynomial in each case, with the given numbers as the sum and product of its zeroes respectively.

   (i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

   (iv) 1, 1 (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) 4, 1

3. Find the quadratic polynomial, for the zeroes α, β given in each case.

   (i) 2, –1 (ii) $\sqrt{3}, -\sqrt{3}$ (iii) $\frac{1}{4}, -1$ (iv) $\frac{1}{2}, \frac{3}{2}$

4. Verify that 1, –1 and –3 are the zeroes of the cubic polynomial $x^3 + 3x^2 - x - 3$ and check the relationship between zeroes and the coefficients.

3.7 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two? For this, let us consider the cubic polynomial $x^3 + 3x^2 - x - 3$. If we tell you that one of its zeroes is 1, then you know that this polynomial is divisible by $x - 1$. Dividing by $x - 1$ we would get the quotient $x^2 - 2x - 3$.

We get the factors of $x^2 - 2x - 3$ by splitting the middle term. The factors are $(x + 1)$ and $(x - 3)$. This gives us

$$x^3 + 3x^2 - x - 3 = (x - 1)(x^2 - 2x - 3)$$

$$= (x - 1)(x + 1)(x - 3)$$

So, the three zeroes of the cubic polynomial are 1, – 1, 3.

Let us discuss the method of dividing one polynomial by another in some detail. Before doing the steps formally, consider a particular example.

Example-8. Divide $2x^2 + 3x + 1$ by $x + 2$.

Solution : Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is $2x - 1$ and the remainder is 3. Also,

$$
\begin{array}{r|l}
 & 2x - 1 \\
\hline
x+2 & 2x^2 + 3x + 1 \\
 & 2x^2 + 4x \\
 & - \quad\ - \\
\hline
 & \quad - x + 1 \\
 & \quad - x - 2 \\
 & \quad + \quad + \\
\hline
 & \qquad\quad 3
\end{array}
$$

$$(2x - 1)(x + 2) + 3 = 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1$$

i.e., $2x^2 + 3x + 1 = (x + 2)(2x - 1) + 3$

Therefore, Dividend = Divisor × Quotient + Remainder

Let us now extend this process to divide a polynomial by a quadratic polynomial.

Example-9. Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.

Solution : We first arrange the terms of the dividend and the divisor in the decreasing order of their degrees. Arranging the terms in this order is termed as writing the polynomials in its standard form. In this example, the dividend is already in its standard form, and the divisor, also in standard form, is $x^2 + 2x + 1$.

$$
\begin{array}{r|l}
 & 3x - 5 \\
\hline
x^2+2x+1 & 3x^3 + x^2 + 2x + 5 \\
 & 3x^3 + 6x^2 + 3x \\
 & - \qquad - \qquad - \\
\hline
 & \quad - 5x^2 - x + 5 \\
 & \quad - 5x^2 - 10x - 5 \\
 & \quad + \qquad + \qquad + \\
\hline
 & \qquad\qquad 9x + 10
\end{array}
$$

Step 1 : To obtain the first term of the quotient, divide the highest degree term of the dividend (i.e., $3x^3$) by the highest degree term of the divisor (i.e., x^2). This is $3x$. Then carry out the division process. What remains is $- 5x^2 - x + 5$.

Step 2 : Now, to obtain the second term of the quotient, divide the highest degree term of the new dividend (i.e., $-5x^2$) by the highest degree term of the divisor (i.e., x^2). This gives -5. Again carry out the division process with $-5x^2 - x + 5$.

Step 3 : What remains is $9x + 10$. Now, the degree of $9x + 10$ is less than the degree of the divisor $x^2 + 2x + 1$. So, we cannot continue the division any further.

So, the quotient is $3x - 5$ and the remainder is $9x + 10$. Also,

$$
\begin{aligned}
(x^2 + 2x + 1) \times (3x - 5) + (9x + 10) &= (3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10) \\
&= 3x^3 + x^2 + 2x + 5
\end{aligned}
$$

Here again, we see that

Dividend = Divisor × Quotient + Remainder

We are applying here an algorithm called Euclid's division algorithm.

This says that

If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x),$$

where either $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$ if $r(x) \neq 0$

This result is known as the **Division Algorithm** for polynomials.

Now, we have the following results from the above discussions

(i) If $q(x)$ is linear polynomial then $r(x) = r$ is a constant.

(ii) If degree of $q(x) = 1$, then degree of $p(x) = 1 +$ degree of $g(x)$.

(iii) If $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ exactly or $q(x)$ is a factor of $p(x)$.

Let us now take some examples to illustrate its use.

Example-10. Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

Solution : Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

So, dividend $= -x^3 + 3x^2 - 3x + 5$ and

divisor $= -x^2 + x - 1$.

Division process is shown on the right side.

$$\begin{array}{r|l} & x - 2 \\ \hline -x^2 + x - 1 & -x^3 + 3x^2 - 3x + 5 \\ & -x^3 + x^2 - x \\ & \;\;+ \quad\;\; - \quad\;\; + \\ \hline & \qquad 2x^2 - 2x + 5 \\ & \qquad 2x^2 - 2x + 2 \\ & \qquad\; - \quad\; + \quad\; - \\ \hline & \qquad\qquad 3 \end{array}$$

We stop here since degree of the remainder is less than the degree of $(-x^2 + x - 1)$ the divisor.

So, quotient $= x - 2$, remainder $= 3$.

Now,

Dividend = Divisor × Quotient + Remainder

$= (-x^2 + x - 1)(x - 2) + 3$

$= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3$

$= -x^3 + 3x^2 - 3x + 5$

In this way, the division algorithm is verified.

Example-11. Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

Solution : Since two of the zeroes are $\sqrt{2}$ and $-\sqrt{2}$, therefore we can divide by $(x-\sqrt{2})(x+\sqrt{2}) = x^2 - 2$.

$$
\begin{array}{r|l}
 & 2x^2 - 3x + 1 \\
\hline
x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\
 & 2x^4 \qquad\quad - 4x^2 \\
 & - \qquad\qquad + \\
\hline
 & -3x^3 + x^2 + 6x - 2 \\
 & -3x^3 \qquad + 6x \\
 & + \qquad\qquad - \\
\hline
 & x^2 - 2 \\
 & x^2 - 2 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

First term of quotient is $\dfrac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\dfrac{-3x^3}{x^2} = -3x$

Third term of quotient is $\dfrac{x^2}{x^2} = 1$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.

Now, by splitting $-3x$, we factorize $2x^2 - 3x + 1$ as $(2x - 1)(x - 1)$. So, its zeroes are given by $x = \dfrac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2}, -\sqrt{2}$, 1 and $\dfrac{1}{2}$.

Exercise – 3.4

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

   (i) $p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

   (ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

   (iii) $p(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$

2. Check in which case the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :

(i) $t^2 - 3,\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$

(ii) $x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$

(iii) $x^3 - 3x + 1,\ x^5 - 4x^3 + x^2 + 3x + 1$

3. Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

4. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.

5. Give examples of polynomials $p(x)$, $g(x)$, $q(x)$ and $r(x)$, which satisfy the division algorithm and

(i) $\deg p(x) = \deg q(x)$ (ii) $\deg q(x) = \deg r(x)$ (iii) $\deg r(x) = 0$

Optional Exercise

[This exercise is not meant for examination]

1. Verify that the numbers given alongside the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

(i) $2x^3 + x^2 - 5x + 2\ ;\ (\frac{1}{2}, 1, -2)$ (ii) $x^3 + 4x^2 + 5x - 2\ ;\ (1, 1, 1)$

2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

3. If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b,\ a,\ a + b$ find a and b.

4. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

5. If the polynomial $x^4 - 6x^3 - 16x^2 + 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

What We Have Discussed

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in x with real coefficients is of the form $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.
3. The zeroes of a polynomial $p(x)$ are the x-coordinates of the points where the graph of $y = p(x)$ intersects the x-axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at the most 3 zeroes.
5. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, $a \neq 0$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

7. The division algorithm states that given any polynomial $p(x)$ and any non-zero polynomial $g(x)$, there are polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x)\, q(x) + r(x),$$