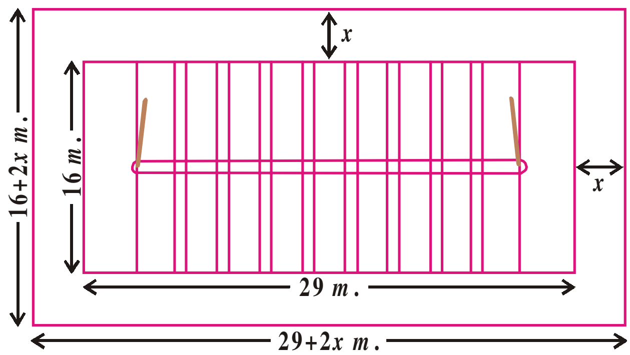
CHAPTER

5 Quadratic Equations

5.1 INTRODUCTION

Sports committee of Kaspa Municipal High School wants to construct a Kho-Kho court of dimension 29 m. × 16 m. This is to be a rectangular enclosure of area 558 m^2. They want to leave space of equal width all around the court for the spectators. What would be the width of the space for spectators? Would it be enough?



Suppose the width of the space be x meter. So from the figure length of the plot would be $(29 + 2x)$ meter.

And, breath of the rectangular plot would be	$= (16 + 2x)\ m.$
Therefore, area of the rectangular plot will be	$=$ length × breadth
	$= (29 + 2x)(16 + 2x)$
Since the area of the plot is	$= 558\ m^2$
$\therefore\ (29 + 2x)(16 + 2x)$	$= 558$
$\therefore\ 4x^2 + 90x + 464$	$= 558$
$4x^2 + 90x - 94$	$= 0$ (dividing by 2)
$2x^2 + 45x - 47$	$= 0$

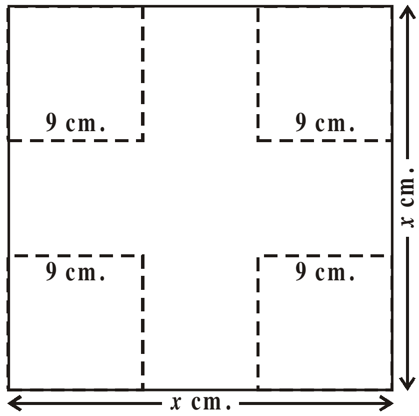
$$2x^2 + 45\,x - 47 = 0 \qquad \text{..... (1)}$$

In previous class we solve the linear equations of the form $ax + b = c$ to find the value of 'x'. Similarly, the value of x from the above equation will give the possible width of the space for spectators.

Can you think of more such examples where we have to find the quantities like in above example and get such equations.

Let us consider another example:

Rani has a square metal sheet. She removed squares of side 9 cm. from each corner of this sheet. Of the remaining sheet, she turned up the sides to form an open box as shown. The capacity of the box is 144 cc. Can we find out the dimensions of the metal sheet?



Suppose the side of the square piece of metal sheet be 'x' *cm*.

Then, the dimensions of the box are

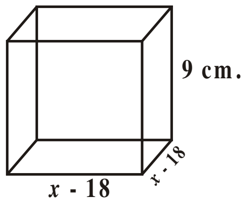
$$9\ cm. \times (x\text{-}18)\ cm. \times (x\text{-}18)\ cm.$$

Since volume of the box is 144 cc

$$9\ (x\text{-}18)\ (x\text{-}18) = 144$$
$$(x\text{-}18)^2 = 16$$
$$x^2 - 36x + 308 = 0$$

So, the side 'x' of the metal sheet will satisfy the equation.

$$x^2 - 36x + 308 = 0 \qquad \ldots\ldots (2)$$



Let us observe the L.H.S of equation (1) and (2)

Are they quadratic polynominals?

We studied such quadratic polynomials of the form $ax^2 + bx + c$, $a \neq 0$ in the previous chapter.

Since, the LHS of the above equations are quadratic polynomials they are called quadratic equations.

In this chapter we will study quadratic equations and methods to find their roots.

5.2 Quadratic Equations

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$. For example, $2x^2 + x - 300 = 0$ is quadratic equation, Similarly, $2x^2 - 3x + 1 = 0$, $4x - 3x^2 + 2 = 0$ and $1 - x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the standard form of a quadratic equation and $y = ax^2 + bx + c$ is called a quadratic function.

Try This

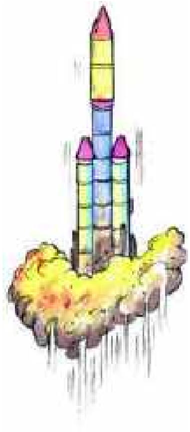
Check whether the following equations are quadratic or not ?

(i) $x^2 - 6x - 4 = 0$ (ii) $x^3 - 6x^2 + 2x - 1 = 0$

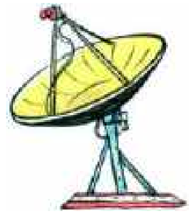
(iii) $7x = 2x^2$ (iv) $x^2 + \dfrac{1}{x^2} = 2$

(v) $(2x + 1)(3x + 1) = b(x - 1)(x - 2)$ (vi) $3y^2 = 192$

There are various uses of Quadratic functions. Some of them are:-



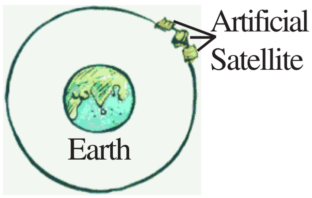
1. When the rocket is fired upward, then the height of the rocket is defined by a 'quadratic function.'
2. Shapes of the satellite dish, reflecting mirror in a telescope, lens of the eye glasses and orbits of the celestial objects are defined by the quadratic equations.

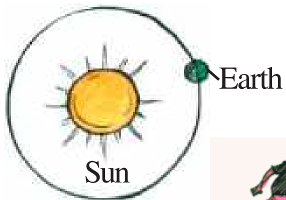


Satellite Dish

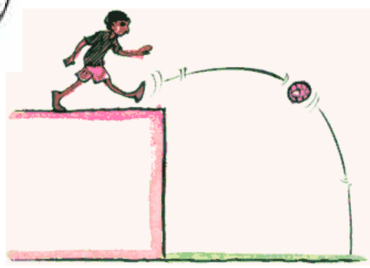
Reflecting Mirror

Lens of Eye Glasses





3. The path of a projectile is defined by quadratic function.



4. When the breaks are applied to a vehicle, the stopping distance is calculated by using quadratic equation.

Example-1. Represent the following situations mathematically:

i. Raju and Rajendar together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles now they have is 124. We would like to find out how many marbles they had previously.

ii. The hypotenuse of a right triangle is 25 *cm*. We know that the difference in lengths of the other two sides is 5 *cm*. We would like to find out the length of the two sides?

Solution : i. Let the number of marbles Raju had be x.

Then the number of marbles Rajendar had = $45 - x$ (Why?).

The number of marbles left with Raju, when he lost 5 marbles = $x - 5$

The number of marbles left with Rajendar, when he lost 5 marbles = $(45 - x) - 5$

$= 40 - x$

Therefore, their product = $(x - 5)(40 - x)$

$= 40x - x^2 - 200 + 5x$

$= -x^2 + 45x - 200$

So, $-x^2 + 45x - 200 = 124$ (Given that product = 124)

i.e., $-x^2 + 45x - 324 = 0$

i.e., $x^2 - 45x + 324 = 0$ (Multiply -ve sign)

Therefore, the number of marbles Raju had 'x', satisfies the quadratic equation

$x^2 - 45x + 324 = 0$

which is the required representation of the problem mathematically.

Let the length of smaller side be x *cm*.

Then length of larger side = $(x + 5)$ *cm*.

Given length of hypotenuse = 25 *cm*.

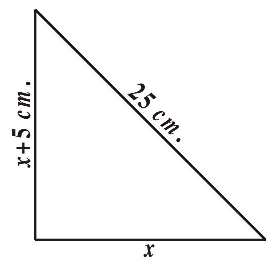
ii. In a right angle triangle we know that $(\text{hypotenuse})^2 = (\text{side})^2 + (\text{side})^2$

So, $x^2 + (x + 5)^2 = (25)^2$

$x^2 + x^2 + 10x + 25 = 625$

$2x^2 + 10x - 600 = 0$

$x^2 + 5x - 300 = 0$



Value of x from the above equation will give the possible value of length of sides of the given right angled triangle.

Example-2. Check whether the following are quadratic equations:

i. $(x - 2)^2 + 1 = 2x - 3$

ii. $x(x + 1) + 8 = (x + 2)(x - 2)$

iii. $x(2x + 3) = x^2 + 1$

iv. $(x + 2)^3 = x^3 - 4$

Solution : i. LHS = $(x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be written as

$x^2 - 4x + 5 = 2x - 3$

i.e., $x^2 - 6x + 8 = 0$

It is in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

ii. Here LHS $= x(x + 1) + 8 = x^2 + x + 8$

and RHS $= (x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

$x^2 + x + 8 - x^2 + 4 = 0$

i.e., $x + 12 = 0$

It is not in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

iii. Here, LHS $= x\,(2x + 3) = 2x^2 + 3x$

So, $x\,(2x + 3) = x^2 + 1$ can be rewritten as

$2x^2 + 3x = x^2 + 1$

Therefore, we get $x^2 + 3x - 1 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

iv. Here, LHS $= (x + 2)^3$ $= (x + 2)^2\,(x + 2)$

$= (x^2 + 4x + 4)\,(x + 2)$

$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8$

$= x^3 + 6x^2 + 12x + 8$

Therefore, $(x + 2)^3 = x^3 - 4$ can be rewritten as

$x^3 + 6x^2 + 12x + 8 = x^3 - 4$

i.e., $6x^2 + 12x + 12 = 0$ or, $x^2 + 2x + 2 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

Remark : In (ii) above, the given equation appears to be a quadratic equation, but it is not a quadratic equation.

In (iv) above, the given equation appears to be a cubic equation (an equation of degree 3) and not a quadratic equation. But it turns out to be a quadratic equation. As you can see, often we need to simplify the given equation before deciding whether it is quadratic or not.

Exercise - 5.1

1. Check whether the following are quadratic equations :

i. $(x + 1)^2 = 2(x - 3)$

ii. $x^2 - 2x = (-2)(3 - x)$

iii. $(x - 2)(x + 1) = (x - 1)(x + 3)$

iv. $(x - 3)(2x + 1) = x(x + 5)$

v. $(2x - 1)(x - 3) = (x + 5)(x - 1)$

vi. $x^2 + 3x + 1 = (x - 2)^2$

vii. $(x + 2)^3 = 2x(x^2 - 1)$

viii. $x^3 - 4x^2 - x + 1 = (x - 2)^3$

2. Represent the following situations in the form of quadratic equations :

i. The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

ii. The product of two consecutive positive integers is 306. We need to find the integers.

iii. Rohan's mother is 26 years older than him. The product of their ages after 3 years will be 360 years. We need to find Rohan's present age.

iv. A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

5.3 Solution of a Quadratic Equation by Factorisation

We have learned to represent some of the daily life situations mathematically in the form of quadratic equation with an unknown variable 'x'.

Now we need to find the value of x.

Consider the quadratic equation $2x^2 - 3x + 1 = 0$. If we replace x by 1. Then, we get $(2 \times 1^2) - (3 \times 1) + 1 = 0$ = RHS of the equation. Since 1 satisfies the equation , we say that 1 is a root of the quadratic equation $2x^2 - 3x + 1 = 0$.

$\therefore$ $x = 1$ is a solution of the quadratic equation.

This also means that 1 is a zero of the quadratic polynomial $2x^2 - 3x + 1$.

In general, a real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. We also say that **$x = \alpha$ is a solution of the quadratic equation**, or **α satisfies the quadratic equation**.

Note that **the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.**

We have observed, in Chapter 3, that a quadratic polynomial can have at most two zeroes. So, any quadratic equation can have atmost two roots. (Why?)

We have learnt in Class-IX, how to factorise quadratic polynomials by splitting their middle terms. We shall use this knowledge for finding the roots of a quadratic equation. Let us see.

Example-3. Find the roots of the equation $2x^2 - 5x + 3 = 0$, by factorisation.

Solution : Let us first split the middle term. Recall that if $ax^2 + bx + c$ is a quadratic equation polynomial then to split the middle term we have to find two numbers p and q such that $p + q = b$ and $p \times q = a \times c$. So to split the middle term of $2x^2 - 5x + 3$, we have to find two numbers p and q such that $p + q = -5$ and $p \times q = 2 \times 3 = 6$.

For this we have to list out all possible pairs of factors of 6. They are (1, 6), (–1, –6); (2, 3); (–2, –3). From the list it is clear that the pair (–2, –3) will satisfy our condition $p + q = -5$ and $p \times q = 6$.

The middle term '$-5x$' can be written as '$-2x - 3x$'.

So, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3 = 2x(x - 1) - 3(x - 1) = (2x - 3)(x - 1)$

Now, $2x^2 - 5x + 3 = 0$ can be rewritten as $(2x - 3)(x - 1) = 0$.

So, the values of x for which $2x^2 - 5x + 3 = 0$ are the same for which $(2x - 3)(x - 1) = 0$,

i.e., either $2x - 3 = 0$ or $x - 1 = 0$.

Now, $2x - 3 = 0$ gives $x = \frac{3}{2}$ and $x - 1 = 0$ gives $x = 1$.

So, $x = \frac{3}{2}$ and $x = 1$ are the solutions of the equation.

In other words, 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Try This

Verify that 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Note that we have found the roots of $2x^2 - 5x + 3 = 0$ by factorising $2x^2 - 5x + 3$ into two linear factors and equating each factor to zero.

Example 4 : Find the roots of the quadratic equation $x - \frac{1}{3x} = \frac{1}{6}$

Solution : We have $x - \frac{1}{3x} = \frac{1}{6} \Rightarrow 6x^2 - x - 2 = 0$

$$6x^2 - x - 2 = 6x^2 + 3x - 4x - 2$$
$$= 3x\,(2x + 1) - 2\,(2x + 1)$$
$$= (3x - 2)(2x + 1)$$

The roots of $6x^2 - x - 2 = 0$ are the values of x for which $(3x - 2)(2x + 1) = 0$

Therefore, $3x - 2 = 0$ or $2x + 1 = 0$,

i.e., $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

Therefore, the roots of $6x^2 - x - 2 = 0$ are $\frac{2}{3}$ and $-\frac{1}{2}$.

We verify the roots, by checking that $\frac{2}{3}$ and $-\frac{1}{2}$ satisfy $6x^2 - x - 2 = 0$.

Example-5. Find the width of the space for spectators discussed in section 5.1.

Solution : In Section 5.1, we found that if the width of the space for spectators is x *m*., then x satisfies the equation $2x^2 + 45x - 47 = 0$. Applying the factorisation method we write this equation as:-

$$2x^2 - 2x + 47x - 47 = 0$$
$$2x\,(x - 1) + 47\,(x - 1) = 0$$

i.e., $(x - 1)\;(2x + 47) = 0$

So, the roots of the given equation are $x = 1$ or $x = \frac{-47}{2}$. Since 'x' is the width of space of the spectators it cannot be negative.

Thus, the width is 1 *m*.

Exercise - 5.2

1. Find the roots of the following quadratic equations by factorisation:

i. $x^2 - 3x - 10 = 0$ ii. $2x^2 + x - 6 = 0$ iii. $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

iv. $2x^2 - x + \frac{1}{8} = 0$ v. $100x^2 - 20x + 1 = 0$ vi. $x(x + 4) = 12$

vii. $3x^2 - 5x + 2 = 0$ viii. $x - \frac{3}{x} = 2$ ix. $3(x - 4)^2 - 5(x - 4) = 12$

2. Find two numbers whose sum is 27 and product is 182.

3. Find two consecutive positive integers, sum of whose squares is 613.

4. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

5. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was Rs 90, find the number of articles produced and the cost of each article.

6. Find the dimensions of a rectangle whose perimeter is 28 meters and whose area is 40 square meters.

7. The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq.cm then find its base and altitude.

8. Two trains leave a railway station at the same time. The first train travels towards west and the second train towards north. The first train travels 5 km/hr faster than the second train. If after two hours they are 50 *km.* apart find the average speed of each train.

9. In a class of 60 students, each boy contributed rupees equal to the number of girls and each girl contributed rupees equal to the number of boys. If the total money then collected was ₹1600. How many boys are there in the class?

10. A motor boat heads upstream a distance of 24km on a river whose current is running at 3 km per hour. The trip up and back takes 6 hours. Assuming that the motor boat maintained a constant speed, what was its speed?

5.4 Solution of a Quadratic Equation by Completing the Square

In the previous section, we have learnt method of factorisation for obtaining the roots of a quadratic equation. Is method of factorization applicable to all types of quadratic equation? Let us try to solve $x^2 + 4x - 4 = 0$ by factorisation method

To solve the given equation $x^2 + 4x - 4 = 0$ by factorization method.

We have to find 'p' and 'q' such that $p + q = 4$ and

$$p \times q = -4$$

But it is not possible. So by factorization method we cannot solve the given equation.

Therefore, we shall study another method.

Consider the following situation

The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?

To answer this, let her present age (in years) be x years. Age before two year = x - 2 & age after four years = $x + 4$ then the product of both the ages is $(x - 2)(x + 4)$.

Therefore, $(x - 2)(x + 4) = 2x + 1$

i.e., $x^2 + 2x - 8 = 2x + 1$

i.e., $x^2 - 9 = 0$

So, Sunita's present age satisfies the quadratic equation $x^2 - 9 = 0$.

We can write this as $x^2 = 9$. Taking square roots, we get $x = 3$ or $x = -3$. Since the age is a positive number, $x = 3$.

So, Sunita's present age is 3 years.

Now consider another quadratic equation $(x + 2)^2 - 9 = 0$. To solve it, we can write it as $(x + 2)^2 = 9$. Taking square roots, we get $x + 2 = 3$ or $x + 2 = -3$.

Therefore, $x = 1$ or $x = -5$

So, the roots of the equation $(x + 2)^2 - 9 = 0$ are 1 and -5.

In both the examples above, the term containing x is completely a square, and we found the roots easily by taking the square roots. But, what happens if we are asked to solve the equation $x^2 + 4x - 4 = 0$. And it cannot be solved by factorisation also.

So, we now introduce the method of completing the square. The idea behind this method is to adjust the left side of the quadratic equation so that it becomes a perfect square.

The process is as follows:

$$x^2 + 4x - 4 = 0$$

$$\Rightarrow \quad x^2 + 4x = 4$$

$$x^2 + 2.\,x\,.\,2 = 4$$

Now, the LHS is in the form of $a^2 + 2ab$. If we add b^2 it becomes as $a^2 + 2ab + b^2$ which is perfect square. So, by adding $b^2 = 2^2 = 4$ to both sides we get,

$$x^2 + 2.x.2 + 2^2 = 4 + 4$$

$$\Rightarrow \quad (x + 2)^2 = 8 \Rightarrow x + 2 = \pm\sqrt{8}$$

$$\Rightarrow \quad x = -2 \pm 2\sqrt{2}$$

Now consider the equation $3x^2 - 5x + 2 = 0$. Note that the coefficient of x^2 is not 1. So we divide the entire equation by 3 so that the coefficient of x^2 is 1

$$\therefore \quad x^2 - \frac{5}{3}x + \frac{2}{3} = 0$$

$$\Rightarrow \quad x^2 - \frac{5}{3}x = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} + \left(\frac{5}{6}\right)^2 = \frac{-2}{3} + \left(\frac{5}{6}\right)^2 \qquad \left(\text{add } \left(\frac{5}{6}\right)^2 \text{ both side}\right)$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-2}{3} + \frac{25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{(12 \times -2) + (25 \times 1)}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-24 + 25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{1}{36} \qquad \text{(take both side square root)}$$

$$x - \frac{5}{6} = \pm\frac{1}{6}$$

So, $x = \frac{5}{6} + \frac{1}{6}$ or $x = \frac{5}{6} - \frac{1}{6}$

Therefore, $x = 1$ or $x = \frac{4}{6}$

i.e., $x = 1$ or $x = \frac{2}{3}$

Therefore, the roots of the given equation are 1 and $\frac{2}{3}$.

From the above examples we can deduce the following algorithm for completing the square.

Algorithm : Let the quadratic equation by $ax^2 + bx + c = 0$

Step-1 : Divide each side by 'a'

Step-2 : Rearrange the equation so that constant term c/a is on the right side. (RHS)

Step-3 : Add $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2$ to both sides to make LHS, a perfect square.

Step-4 : Write the LHS as a square and simplify the RHS.

Step-5 : Solve it.

Example-6. Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.

Solution : Given : $5x^2 - 6x - 2 = 0$

Now we follow the Algoritm

Step-1 : $x^2 - \frac{6}{5}x - \frac{2}{5} = 0$ (Dividing both sides by 5)

Step-2 : $x^2 - \frac{6}{5}x = \frac{2}{5}$

Step-3 : $x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = \frac{2}{5} + \left(\frac{3}{5}\right)^2$ $\left(\text{Adding } \left(\frac{3}{5}\right)^2 \text{ to both sides}\right)$

Step-4 : $\left(x - \frac{3}{5}\right)^2 = \frac{2}{5} + \frac{9}{25}$

Step-5 : $\left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$

$$x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}}$$

$$x = \frac{3}{5} + \frac{\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3}{5} - \frac{\sqrt{19}}{5}$$

$$\therefore \quad x = \frac{3+\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3-\sqrt{19}}{5}$$

Example-7. Find the roots of $4x^2+3x+5=0$ by the method of completing the square.

Solution : Given $4x^2 + 3x + 5 = 0$

$$x^2+\frac{3}{4}x+\frac{5}{4}=0$$

$$x^2+\frac{3}{4}x=\frac{-5}{4}$$

$$x^2+\frac{3}{4}x+\left(\frac{3}{8}\right)^2=\frac{-5}{4}+\left(\frac{3}{8}\right)^2$$

$$\left(x+\frac{3}{8}\right)^2=\frac{-5}{4}+\frac{9}{64}$$

$$\left(x+\frac{3}{8}\right)^2=\frac{-71}{64}<0$$

But $\left(x+\frac{3}{8}\right)^2$ cannot be negative for any real value of x (Why?). So, there is no real value of x satisfying the given equation. Therefore, the given equation has no real roots.

Do This

Solve the equations by completing the square

(i) $x^2 - 10x + 9 = 0$

(ii) $x^2 - 5x + 5= 0$

(iii) $x^2 + 7x - 6 = 0$

We have solved several examples with the use of the method of 'completing the square.' Now, let us apply this method in standard form of quadratic equation $ax^2+bx+c=0$ ($a\neq0$).

Step 1 : Dividing the equation through out by 'a' we get

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Step 2 : $x^2+\frac{b}{a}x=-\frac{c}{a}$

Step 3 : $x^2 + \frac{b}{a}x + \left[\frac{1}{2}\frac{b}{a}\right]^2 = -\frac{c}{a} + \left[\frac{1}{2}\frac{b}{a}\right]^2$ $\left[\text{adding } \left[\frac{1}{2}\frac{b}{a}\right]^2 \text{ both sides}\right]$

$$\Rightarrow x^2 + 2 \cdot x \frac{b}{2a} + \left[\frac{b}{2a}\right]^2 = -\frac{c}{a} + \left[\frac{b}{2a}\right]^2$$

Step 4 : $\left[x + \frac{b}{2a}\right]^2 = \frac{b^2 - 4ac}{4a^2}$

Step 5 : If $b^2 - 4ac \geq 0$, then by taking the square roots, we get

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Therefore, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So, the roots of $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$, if $b^2 - 4ac \geq 0$.

If $b^2 - 4ac < 0$, the equation will have no real roots. (Why?)

Thus, if $b^2 - 4ac \geq 0$, then the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This formula for finding the roots of a quadratic equation is known as the **quadratic formula**.

Let us consider some examples by using quadratic formula.

Example-8. Solve Q. 2(i) of Exercise 5.1 by using the quadratic formula.

Solution : Let the breadth of the plot be x metres.

Then the length is $(2x + 1)$ metres.

Since area of rectangular plot is 528 m^2

We can write $x(2x + 1) = 528$, i.e., $2x^2 + x - 528 = 0$.

This is in the form of $ax^2 + bx + c = 0$, where $a = 2$, $b = 1$, $c = -528$.

So, the quadratic formula gives us the solution as

$$x = \frac{-1 \pm \sqrt{1+4(2)(528)}}{4} = \frac{-1 \pm \sqrt{4225}}{4} = \frac{-1 \pm 65}{4}$$

i.e., $x = \frac{64}{4}$ or $x = \frac{-66}{4}$

i.e., $x = 16$ or $x = -\frac{33}{2}$

Since x cannot be negative. So, the breadth of the plot is 16 metres and hence, the length of the plot is $(2x + 1) = 33$m.

You should verify that these values satisfy the conditions of the problem.

THINK - DISCUSS

We have three methods to solve a quadratic equation. Among these three, which method would you like to use? Why?

Example-9. Find two consecutive odd positive integers, sum of whose squares is 290.

Solution : Let first odd positive integers be x. Then, the second integer will be $x + 2$. According to the question,

$$x^2 + (x + 2)^2 = 290$$

i.e., $x^2 + x^2 + 4x + 4 = 290$

i.e., $2x^2 + 4x - 286 = 0$

i.e., $x^2 + 2x - 143 = 0$

which is a quadratic equation in x.

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

we get, $x = \frac{-2 \pm \sqrt{4+572}}{2} = \frac{-2 \pm \sqrt{576}}{2} = \frac{-2 \pm 24}{2}$

i.e., $x = 11$ or $x = -13$

But x is given to be an odd positive integer. Therefore, $x \neq -13$, $x = 11$.

Thus, the two consecutive odd integers are 11 and $(x + 2) = 11 + 2 = 13$.

Check : $11^2 + 13^2 = 121 + 169 = 290$.

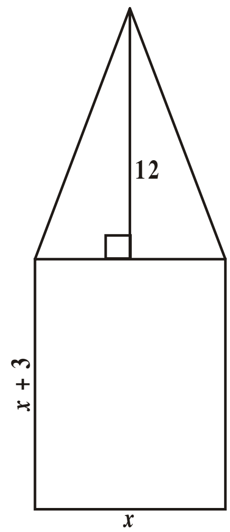
Example-10. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (see Fig. 5.3). Find its length and breadth.

Solution : Let the breadth of the rectangular park be x m.

So, its length = $(x + 3)$ m.

Therefore, the area of the rectangular park = $x(x + 3)\text{ m}^2 = (x^2 + 3x)\text{ m}^2$.

Now, base of the isosceles triangle = x m.



Therefore, its area = $\frac{1}{2} \times x \times 12 = 6\,x\text{ m}^2$.

According to our requirements,

$$x^2 + 3x = 6x + 4$$

i.e., $$x^2 - 3x - 4 = 0$$

Using the quadratic formula, we get

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \; = 4 \text{ or } -1$$

But $x \neq -1$ (Why?). Therefore, $x = 4$.

So, the breadth of the park = 4m and its length will be $x + 3 = 4 + 3 = 7$m.

Verification : Area of rectangular park = 28 m^2,

area of triangular park = 24 m^2 = $(28 - 4)\text{ m}^2$

Example-11. Find the roots of the following quadratic equations, if they exist, using the quadratic formula:

(i) $x^2 + 4x + 5 = 0$ (ii) $2x^2 - 2\sqrt{2x} + 1 = 0$

Solution :

(i) $x^2 + 4x + 5 = 0$. Here, $a = 1$, $b = 4$, $c = 5$. So, $b^2 - 4ac = 16 - 20 = -4 < 0$.

Since the square of a real number cannot be negative, therefore $\sqrt{b^2 - 4ac}$ will not have any real value.

So, there are no real roots for the given equation.

(ii) $2x^2 - 2\sqrt{2}\,x + 1 = 0$. Here, $a = 2$, $b = -2\sqrt{2}$, $c = 1$.

So, $b^2 - 4ac = 8 - 8 = 0$

Therefore, $x = \dfrac{2\sqrt{2} \pm \sqrt{0}}{4} = \dfrac{\sqrt{2}}{2} \pm 0$ *i.e.,* $x = \dfrac{1}{\sqrt{2}}$.

So, the roots are $\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$.

Example-12. Find the roots of the following equations:

(i) $x + \dfrac{1}{x} = 3,\ x \neq 0$ (ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0, 2$

Solution :

(i) $x + \dfrac{1}{x} = 3$. Multiplying whole by x, we get

$x^2 + 1 = 3x$

i.e., $x^2 - 3x + 1 = 0$, which is a quadratic equation.

Here, $a = 1, b = -3, c = 1$

So, $b^2 - 4ac = 9 - 4 = 5 > 0$

Therefore, $x = \dfrac{3 \pm \sqrt{5}}{2}$ (why ?)

So, the roots are $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$.

(ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0,\ 2.$

As $x \neq 0, 2$, multiplying the equation by $x\,(x-2)$, we get

$(x - 2) - x = 3x\,(x - 2)$

$= 3x^2 - 6x$

So, the given equation reduces to $3x^2 - 6x + 2 = 0$, which is a quadratic equation.

Here, $a = 3, b = -6, c = 2$. So, $b^2 - 4ac = 36 - 24 = 12 > 0$

Therefore, $x = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = \dfrac{3 \pm \sqrt{3}}{3}$.

So, the roots are $\frac{3+\sqrt{3}}{3}$ and $\frac{3-\sqrt{3}}{3}$.

Example-13. A motor boat whose speed is 18 km/h in still water. It takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

Solution : Let the speed of the stream be x km/h.

Therefore, the speed of the boat upstream = $(18 - x)$ km/h and the speed of the boat downstream = $(18 + x)$ km/h.

The time taken to go upstream = $\frac{\text{distance}}{\text{speed}} = \frac{24}{18-x}$ hours.

Similarly, the time taken to go downstream = $\frac{24}{18+x}$ hours.

According to the question,

$$\frac{24}{18-x} - \frac{24}{18+x} = 1$$

i.e., $24(18 + x) - 24(18 - x) = (18 - x)(18 + x)$

i.e., $x^2 + 48x - 324 = 0$

Using the quadratic formula, we get

$$x = \frac{-48 \pm \sqrt{48^2 + 1296}}{2} = \frac{-48 \pm \sqrt{3600}}{2}$$

$$= \frac{-48 \pm 60}{2} = 6 \text{ or } -54$$

Since x is the speed of the stream, it cannot be negative. So, we ignore the root $x = -54$. Therefore, $x = 6$ gives the speed of the stream as 6 km/h.

Exercise - 5.3

1. Find the roots of the following quadratic equations, if they exist, by the method of completing the square:

i. $2x^2 + x - 4 = 0$

ii. $4x^2 + 4\sqrt{3}x + 3 = 0$

iii. $5x^2 - 7x - 6 = 0$

iv. $x^2 + 5 = -6x$

2. Find the roots of the quadratic equations given in Q.1 above by applying the quadratic formula.

3. Find the roots of the following equations:

(i) $x - \frac{1}{x} = 3, \ x \neq 0$ (ii) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, \ x \neq -4, 7$

4. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

5. In a class test, the sum of Moulika's marks in Mathematics and English is 30. If she got 2 marks more in Mathematics and 3 marks less in English, the product of her marks would have been 210. Find her marks in the two subjects.

6. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

7. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

8. A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

10. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11km/h more than that of the passenger train, find the average speed of the two trains.

11. Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, find the sides of the two squares.

12. A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity 80 m/second. The distance 's' of the ball from the ground after t seconds is S = $96 + 80t - 16t^2$. After how may seconds does the ball strike the ground.

13. If a polygon of 'n' sides has $\frac{1}{2} n (n-3)$ diagonals. How many sides will a polygon having 65 diagonals? Is there a polygon with 50 diagonals?

5.5 Nature of Roots

In the previous section, we have seen that the roots of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now let us try to understand the nature of roots.

Remember that zeros are those points where value of polynomial becomes zero or we can say that the curve of quadratic polynomial cuts the X-axis.

Similarly, roots of a quadratic equation are those points where the curve cuts the X-axis.

Case-1 : If $b^2 - 4ac > 0$;

We get two distinct real roots $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

In such case if we draw graph for the given quadratic equation we get the following figures.

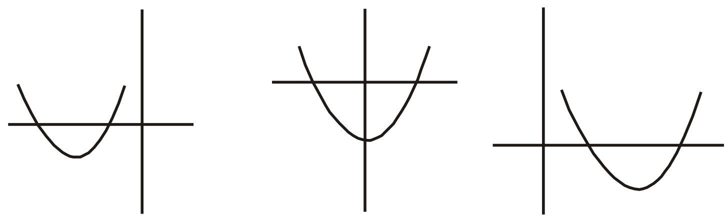


Figure shows that the curve of the quadratic equation cuts the x-axis at two distinct points

Case-2 : If $b^2 - 4ac = 0$

$$x = \frac{-b + 0}{2a}$$

So, $x = \frac{-b}{2a}, \frac{-b}{2a}$

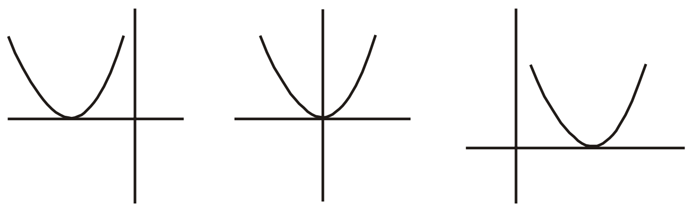
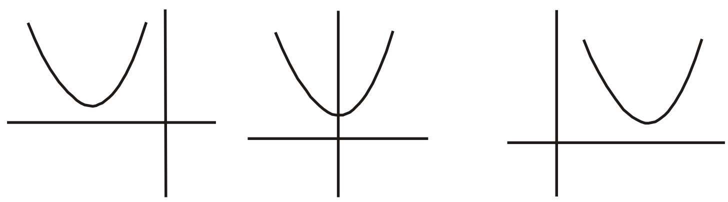


Figure shows that the curve of the quadratic equation touching X-axis at one point.

Case-3 : $b^2 - 4ac < 0$

There are no real roots. Roots are imaginary.



In this case graph neither intersects nor touches the X-axis at all. So, there are no real roots.

Since $b^2 - 4ac$ determines whether the quadratic equation $ax^2 + bx + c = 0$ has real roots or not, $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

So, a quadratic equation $ax^2 + bx + c = 0$ has

i. two distinct real roots, if $b^2 - 4ac > 0$,

ii. two equal real roots, if $b^2 - 4ac = 0$,

iii. no real roots, if $b^2 - 4ac < 0$.

Let us consider some examples.

Example-14. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

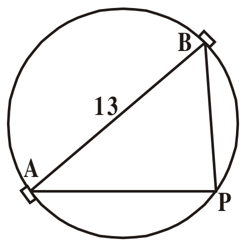
Solution : The given equation is in the form of $ax^2 + bx + c = 0$, where $a = 2$, $b = -4$ and $c = 3$. Therefore, the discriminant

$$b^2 - 4ac = (-4)^2 - (4 \times 2 \times 3) = 16 - 24 = -8 < 0$$

So, the given equation has no real roots.

Example-15. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Solution : Let us first draw the diagram.



Let P be the required location of the pole. Let the distance of the pole from the gate B be x m, i.e., BP $= x$ m. Now the difference of the distances of the pole from the two gates = AP – BP (or, BP – AP)= 7 m. Therefore, AP $= (x + 7)$ m.

Now, AB = 13m, and since AB is a diameter,

$\angle APB = 90^0$ (Why?)

Therefore, $AP^2 + PB^2 = AB^2$ (By Pythagoras theorem)

i.e., $(x + 7)^2 + x^2 = 13^2$

i.e., $x^2 + 14x + 49 + x^2 = 169$

i.e., $2x^2 + 14x - 120 = 0$

So, the distance 'x' of the pole from gate B satisfies the equation

$$x^2 + 7x - 60 = 0$$

So, it would be possible to place the pole if this equation has real roots. To see if this is so or not, let us consider its discriminant. The discriminant is

$$b^2 - 4ac = 7^2 - 4 \times 1 \times (-60) = 289 > 0.$$

So, the given quadratic equation has two real roots, and it is possible to erect the pole on the boundary of the park.

Solving the quadratic equation $x^2 + 7x - 60 = 0$, by the quadratic formula, we get

$$x = \frac{-7 \pm \sqrt{289}}{2} = \frac{-7 \pm 17}{2}$$

Therefore, $x = 5$ or -12.

Since x is the distance between the pole and the gate B, it must be positive.

Therefore, $x = -12$ will have to be ignored. So, $x = 5$.

Thus, the pole has to be erected on the boundary of the park at a distance of 5m from the gate B and 12m from the gate A.

Try This

1. Explain the benefits of evaluating the discriminant of a quadritic equation before attempting to solve it. What does its value signifies?
2. Write three quadratic equations one having two distinct real solutions, one having no real solution and one having exactly one real solution.

Example-16. Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

Solution : Here $a = 3$, $b = -2$ and $c = \frac{1}{3}$

Therefore, discriminant $b^2 - 4ac = (-2)^2 - 4 \times 3 \times \frac{1}{3} = 4 - 4 = 0.$

Hence, the given quadratic equation has two equal real roots.

The roots are $\frac{-b}{2a}, \frac{-b}{2a}$, i.e., $\frac{2}{6}, \frac{2}{6}$, i.e., $\frac{1}{3}, \frac{1}{3}$.

Exercise - 5.4

1. Find the nature of the roots of the following quadratic equations. If real roots exist, find them:

   (i) $2x^2 - 3x + 5 = 0$ (ii) $3x^2 - 4\sqrt{3}x + 4 = 0$

   (iii) $2x^2 - 6x + 3 = 0$

2. Find the values of k for each of the following quadratic equations, so that they have two equal roots.

   (i) $2x^2 + kx + 3 = 0$ (ii) $kx\,(x - 2) + 6 = 0$

3. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.

4. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48. Is the situation possible? If so, determine their present ages.

5. Is it possible to design a rectangular park of perimeter 80 m. and area 400 m^2? If so, find its length and breadth.

Optional Exercise

[This exercise is not meant for examination]

1. Some points are plotted on a plane. Each point joined with all remaining points by line segments. Find the number of points if the number of line segments are 10.

2. A two digit number is such that the product of the digits is 8. When 18 is added to the number they interchange their places. Determine the number.

3. A piece of wire 8 m. in length, cut into two pieces, and each piece is bent into a square. Where should the cut in the wire be made if the sum of the areas of these squares is to be 2 m^2?

   $$\left[\text{Hint} : x + y = 8, \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 2 \Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{8-x}{4}\right)^2 = 2\right].$$

4. Vinay and Praveen working together can paint the exterior of a house in 6 days. Vinay by himself can complete the job in 5 days less than Praveen. How long will it take vinay to complete the job by himself.

5. Show that the sum of roots of a quadratic equation is '$\frac{-b}{a}$'.

6. Show that the product of the roots of a quadratic equation is '$\frac{c}{a}$'.

7. The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.

What We Have Discussed

In this chapter, we have studied the following points:

1. Standard form of quadratic equation in variable x is $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

3. If we can factorise $ax^2 + bx + c$, $a \neq 0$, into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.

4. A quadratic equation can also be solved by the method of completing the square.

5. Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided } b^2 - 4ac \geq 0.$$

6. A quadratic equation $ax^2 + bx + c = 0$ has

   (i) two distinct real roots, if $b^2 - 4ac > 0$,

   (ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

   (iii) no real roots, if $b^2 - 4ac < 0$.