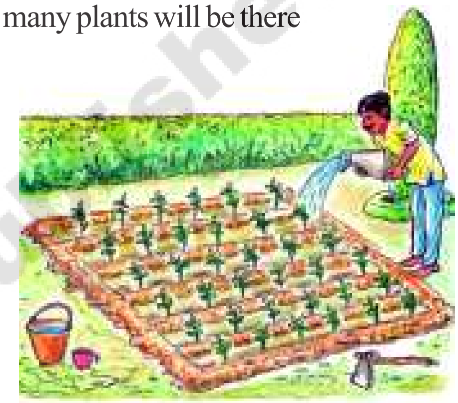
Polynomials and Factorisation

02

2.1 Introduction

There are six rows and each row has six plants in a garden. How many plants are there in total ? If there are 'x' plants, planted in 'x' rows then how many plants will be there in the garden? Obviously it is x^2.



The cost of onions is ₹10 per kg. Inder purchased p kg., Raju purchased q kg. and Hanif purchased r kg. How much each would have paid? The payments would be ₹10p, ₹10q and ₹10r respectively. *All such examples show the use of algebraic expression.*

We also use algebraic expressions such as 's^2' to find area of a square, 'lb' for area of a rectangle and 'lbh' for volume of a cuboid. What are the other algebraic expressions that we use?

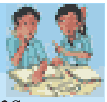
Algebraic expressions such as $3xy$, x^2+2x, $x^3 - x^2 + 4x + 3$, πr^2, $ax + b$ etc. are called polynomials. Note that, all algebraic expressions we have considered so far only have non-negative integers as exponents of the variables.

Can you find the polynomials among the given algebraic expressions:

$$x^2, \quad x^{\frac{1}{2}} + 3, \quad 2x^2 - \frac{3}{x} + 5; \quad x^2 + xy + y^2$$

From the above $x^{\frac{1}{2}} + 3$ is not a polynomials because the first term $x^{\frac{1}{2}}$ is a term with an exponent that is not a non-negative integer (i.e. $\frac{1}{2}$) and also $2x^2 - \frac{3}{x} + 5$ is not a polynomial because it can be written as $2x^2 - 3x^{-1} + 5$. Here the second term ($3x^{-1}$) has a negative exponent. (i.e., -1). An algebraic expression in which the variables involved have only non-negative integral powers is called a polynomial.

THINK, DISCUSS AND WRITE



Which of the following expressions are polynomials ? Which are not ? Give reasons.

(i) $4x^2 + 5x - 2$ (ii) $y^2 - 8$ (iii) 5 (iv) $2x^2 + \frac{3}{x} - 5$

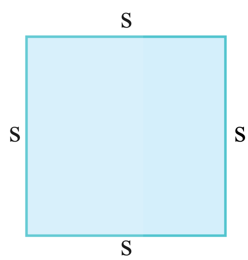
(v) $\sqrt{3}x^2 + 5y$ (vi) $\frac{1}{x+1}$ (vii) $\sqrt{x}$ (viii) $3\,xyz$

We shall start our study with polynomials in various forms. In this chapter we will also learn factorisation of polynomials using Remainder Theorem and Factor Theorem and their use in the factorisation of polynomials.

2.2 POLYNOMIALS IN ONE VARIABLE

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z etc. to denote variables. We have algebraic expressions

$2x, 3x, -x, \frac{3}{4}x$ .... all in one variable x. These expressions are of the form (a constant) × (some power of variable). Now, suppose we want to find the perimeter of a square we use the formula P = 4s.



Here '4' is a constant and 's' is a variable, representing the side of a square. The side could vary for different squares.

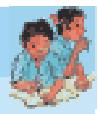
Observe the following table:

Side of square (s)	Perimeter (4s)
4 cm	P = 4 × 4 = 16 cm
5 cm	P = 4 × 5 = 20 cm
10 cm	P = 4 × 10 = 40 cm

Here the value of the constant i.e. '4' remains the same throughout this situation. That is, the value of the constant does not change in a given problem, but the value of the variable (s) keeps changing.

Suppose we want to write an expression which is of the form '(a constant) × (a variable)' and we do not know, what the constant is, then we write the constants as a, b, c ... etc. So these expressions in general will be ax, by, cz, ... etc. Here a, b, c ... are arbitrary constants. You are also familiar with other algebraic expressions like x^2, $x^2 + 2x + 1$, $x^3 + 3x^2 - 4x + 5$. All these expressions are polynomials in one variable.

DO THESE



- Write two polynomials with variable 'x'
- Write three polynomials with variable 'y'
- Is the polynomial $2x^2 + 3xy + 5y^2$ in one variable ?
- Write the formulae of area and volume of different solid shapes. Find out the variables and constants in them.

2.3 DEGREE OF THE POLYNOMIAL

Each term of the polynomial consists of the products of a constant, called the **coefficient** of the term and a finite number of variables raised to non-negative integral powers. Degree of a term is the sum of the exponent of is variable factors. And the degree of a polynomial is the largest degree of its variable term.

Lets find the terms, their coefficients and the degree of polynomials:

(i) $3x^2 + 7x + 5$ (ii) $3x^2y^2 + 4xy + 7$

In the polynomial $3x^2 + 7x + 5$, each of the expressions $3x^2$, $7x$ and 5 are terms. Each term of the polynomial has a coefficient, so in $3x^2 + 7x + 5$, the coefficient of x^2 is 3, the coefficient of x is 7 and 5 is the coefficient of x^0 (Remember x^0=1)

You know that the degree of a polynomial is the highest degree of its variable term.

As the term $3x^2$ has the highest degree among all the other terms in that expression, Thus the degree of $3x^2 + 7x + 5$ is '2'.

Now can you find coefficient and degree of polynomial $3x^2y^3 + 4xy + 7$.

The coefficient of x^2y^3 is 3, xy is 4 and x^0y^0 is 7. The sum of the exponents of the variables in term $3x^2y^3$ is $2 + 3 = 5$ which is greater than that of the other terms. So the degree of polynomial $3x^2y^3 + 4xy + 7$ is 5.

Now think what is the degree of a constant? As the constant contains no variable, it can be written as product of x^0. For example, degree of 5 is zero as it can be written as $5x^0$. Now that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (i.e. all the coefficients are zero), we get the **zero polynomial**, which is the number '0'.

Can you say the degree of zero? It is not defined as we can't write it as a product of a variable raised to any power.

DO THESE

1. Write the degree of each of the following polynomials
   (i) $7x^3 + 5x^2 + 2x - 6$ (ii) $7 - x + 3x^2$
   (iii) $5p - \sqrt{3}$ (iv) 2 (v) $-5xy^2$
2. Write the coefficient of x^2 in each of the following
   (i) $15 - 3x + 2x^2$ (ii) $1 - x^2$ (iii) $\pi x^2 - 3x + 5$ (iv) $\sqrt{2}x^2 + 5x - 1$

Let us observe the following tables and fill the blanks.

(i) Types of polynomials according to degree :

Degree of a polynomial	Name of the polynomial	Example
Not defined	Zero polynomial	0
Zero	Constant polynomial	$-12;\ 5;\ \frac{3}{4}$ etc
1	....	$x - 12;\ -7x + 8;\ ax + b$ etc.
2	Quadratic polynomial	....
3	Cubic polynomial	$3x^3 - 2x^2 + 5x + 7$

Usually, a polynomial of degree 'n' is called n^{th} degree polynomial.

(ii) Types of polynomials according to number of terms:

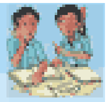
Number of non - zero terms	Name of the polynomial	Example	Terms
1	Monomial	$-3x$	$-3x$
2	Binomial	$3x + 5$	$3x$, 5
3	Trinomial	$2x^2 + 5x + 1$	....
More than 3	Multinomial	....	$3x^3$, $2x^2$, $-7x$, 5

Note : Every polynomial is a multinomial but every multinomial need not be a polynomial.

A linear polynomial with one variable may be a monomial or a binomial.

Eg : $3x$ or $2x - 5$

THINK, DISCUSS AND WRITE



How many terms a cubic (degree 3) polynomial with one variable can have? Give examples.

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, $q(x)$ or $r(x)$ etc. So for example, we may write

$p(x) = 3x^2 + 2x + 1$

$q(x) = x^3 - 5x^2 + x - 7$

$r(y) = y^4 - 1$

$t(z) = z^2 + 5z + 3$

TRY THESE

1. Write a polynomial with 2 terms in variable x.
2. How can you write a polynomial with 15 terms in variable p?

A polynomial can have any finite number of terms.

So far mostly we have discussed the polynomials in one variable only. We can also have polynomials in more than one variable. For example $x + y$, $x^2 + 2xy + y^2$, $x^2 - y^2$ are polynomials in two variables x, y. Similarly $x^2 + y^2 + z^2$, $x^3 + y^3 + z^3$ are polynomials in three variables. You will study such polynomials later in detail.

EXERCISE - 2.1

1. Find the degree of each of the polynomials given below

(i) $x^5 - x^4 + 3$ (ii) $x^2 + x - 5$ (iii) 5

(iv) $3x^6 + 6y^3 - 7$ (v) $4 - y^2$ (vi) $5t - \sqrt{3}$

2. Which of the following expressions are polynomials in one variable and which are not ? Give reasons for your answer.

(i) $3x^2 - 2x + 5$ (ii) $x^2 + \sqrt{2}$ (iii) $p^2 - 3p + q$ (iv) $y + \frac{2}{y}$

(v) $5\sqrt{x} + x\sqrt{5}$ (vi) $x^{100} + y^{100}$

3. Write the coefficient of x^3 in each of the following

(i) $x^3 + x + 1$ (ii) $2 - x^3 + x^2$ (iii) $\sqrt{2}x^3 + 5$ (iv) $2x^3 + 5$

(v) $\frac{\pi}{2}x^3 + x$ (vi) $-\frac{2}{3}x^3$ (vii) $2x^2 + 5$ (vi) 4

4. Classify the following as linear, quadratic and cubic polynomials

(i) $5x^2 + x - 7$ (ii) $x - x^3$ (iii) $x^2 + x + 4$ (iv) $x - 1$

(v) $3p$ (vi) πr^2

5. Write whether the following statements are True or False. Justify your answer

(i) A binomial can have at the most two terms

(ii) Every polynomial is a binomial

(iii) A binomial may have degree 3

(iv) Degree of zero polynomial is zero

(v) The degree of $x^2 + 2xy + y^2$ is 2

(vi) πr^2 is monomial.

6. Give one example each of a monomial and trinomial of degree 10.

2.4 ZEROES OF A POLYNOMIAL

- Consider the polynomial $p(x) = x^2 + 5x + 4$.

  What is the value of $p(x)$ for $x = 1$.

  For this we have to replace x by 1 every where in $p(x)$

By doing this $p(1) = (1)^2 + 5(1) + 4,$

we get $= 1 + 5 + 4 = 10$

So, we say that the value of $p(x)$ at $x = 1$ is 10

Similarly find $p(x)$ for $x = 0$ and $x = -1$

$$p(0) = (0)^2 + 5(0) + 4 = 0 + 0 + 4 = 4$$

$$p(-1) = (-1)^2 + 5(-1) + 4 = 1 - 5 + 4 = 0$$

Can you find the value of $p(-4)$?

- Consider another polynomial

$$s(y) = 4y^4 - 5y^3 - y^2 + 6$$

$$\begin{aligned} s(1) &= 4(1)^4 - 5(1)^3 - (1)^2 + 6 \\ &= 4(1) - 5(1) - 1 + 6 \\ &= 4 - 5 - 1 + 6 \\ &= 10 - 6 \\ &= 4 \end{aligned}$$

Can you find $s(-1) = ?$

DO THIS

Find the value of each of the following polynomials for the indicated value of variables:

(i) $p(x) = 4x^2 - 3x + 7$ at $x = 1$

(ii) $q(y) = 2y^3 - 4y + \sqrt{11}$ at $y = 1$

(iii) $r(t) = 4t^4 + 3t^3 - t^2 + 6$ at $t = p,\ t \in R$

(iv) $s(z) = z^3 - 1$ at $z = 1$

(v) $p(x) = 3x^2 + 5x - 7$ at $x = 1$

(vi) $q(z) = 5z^3 - 4z + \sqrt{2}$ at $z = 2$

- Now consider the polynomial $r(t) = t - 1$

What is $r(1)$? It is $r(1) = 1 - 1 = 0$

As $r(1) = 0$, we say that 1 is a zero of the polynomial $r(t)$.

In general, we say that a **zero of a polynomial** $p(x)$ is the value of x, for which $p(x) = 0$.

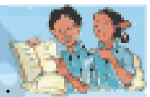
This value is also called a root of the polynomial $p(x)$

What is the zero of polynomial $f(x) = x + 1$?

You must have observed that the zero of the polynomial $x + 1$ is obtained by equating it to 0. i.e., $x + 1 = 0$, which gives $x = -1$. If $f(x)$ is a polynomial in x then $f(x) = 0$ is called a polynomial equation in x . We observe that '-1' is the root of the polynomial $f(x)$ in the above example So we say that '-1' is the zero of the polynomial $x + 1$, or a root of the polynomial equation $x + 1 = 0$.

- Now, consider the constant polynomial 3. Can you tell what is its zero ? It does not have a zero. As $3 = 3x^0$ no real value of x gives value of $3x^0$. Thus a constant polynomial has no zeroes. But zero polynomial is a constant polynomial having many zeros.

TRY THESE



Find zeroes of the following polynomials

1. $2x - 3$
2. $x^2 - 5x + 6$
3. $x + 5$

Example-1. $p(x) = x + 2$. Find $p(1)$, $p(2)$, $p(-1)$ and $p(-2)$. What are zeroes of the polynomial $x + 2$?

Solution : Let $p(x) = x + 2$

replace x by 1

$p(1) = 1 + 2 = 3$

replace x by 2

$p(2) = 2 + 2 = 4$

replace x by -1

$p(-1) = -1 + 2 = 1$

replace x by -2

$p(-2) = -2 + 2 = 0$

Therefore, 1 , 2, -1 are not the zeroes of the polynomial $x + 2$, but -2 is the zero of the polynomial.

Example-2. Find a zero of the polynomial $p(x) = 3x + 1$

Solution : Finding a zero of $p(x)$, is same as solving the equation

$$p(x) = 0$$

i.e.

$$3x + 1 = 0$$

$$3x = -1$$

$$x = -\frac{1}{3}$$

So, $-\frac{1}{3}$ is a zero of the polynomial $3x+1$.

Example-3. Find a zero of the polynomial $2x-1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation $p(x)=0$

As $2x - 1 = 0$

$$x = \frac{1}{2} \text{ (how ?)}$$

Check it by finding the value of $P\left(\frac{1}{2}\right)$

Now, if $p(x) = ax+b$, $a \neq 0$, a linear polynomial, how will you find a zero of $p(x)$?

As we have seen to find zero of a polynomial $p(x)$, we need to solve the polynomial equation $p(x)=0$

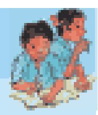
Which means $ax+b=0$, $a \neq 0$

So $ax = -b$

i.e., $x = \frac{-b}{a}$

So, $x = \frac{-b}{a}$ is the only zero of the polynomial $p(x)=ax+b$ i.e., **A linear polynomial in one variable has only one zero.**

DO THIS



Fill in the blanks:

Linear Polynomial	Zero of the polynomial
$x + a$	$-a$
$x - a$	-------------
$ax + b$	-------------
$ax - b$	$\frac{b}{a}$

Example-4. Verify whether 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$

Solution : Let $p(x) = x^2 - 3x + 2$

replace x by 2

$$p(2) = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0$$

also replace x by 1

$$p(1) = (1)^2 - 3(1) + 2 = 1 - 3 + 2 = 0$$

Hence, both 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$.

Is there any other way of checking this?

What is the degree of the polynomial $x^2 - 3x + 2$? Is it a linear polynomial ? No, it is a quadratic polynomial. Hence, a quadratic polynomial has two zeroes.

Example-5. If 3 is a zero of the polynomial $x^2 + 2x - a$. Find a?

Solution : Let $p(x) = x^2 + 2x - a$

As the zero of this polynomial is 3, we know that $p(3) = 0$.

$$x^2 + 2x - a = 0$$

Put $x = 3$, $(3)^2 + 2(3) - a = 0$

$$9 + 6 - a = 0$$

$$15 - a = 0$$

$$-a = -15$$

or $a = 15$

THINK AND DISCUSS

1. $x^2 + 1$ has no zeros. Why?
2. Can you tell the number of zeroes a polynomial of degree 'n' will have?

EXERCISE - 2.2

1. Find the value of the polynomial $4x^2 - 5x + 3$, when

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$ (iv) $x = \frac{1}{2}$

2. Find $p(0), p(1)$ and $p(2)$ for each of the following polynomials.

(i) $p(x) = x^2 - x + 1$ (ii) $p(y) = 2 + y + 2y^2 - y^3$

(iii) $p(z) = z^3$ (iv) $p(t) = (t-1)(t+1)$

(v) $p(x) = x^2 - 3x + 2$

3. Verify whether the values of x given in each case are the zeroes of the polynomial or not?.

(i) $p(x) = 2x + 1;\ x = -\dfrac{1}{2}$ (ii) $p(x) = 5x - \pi;\ x = \dfrac{-3}{2}$

(iii) $p(x) = x^2 - 1;\ x = \pm 1$ (iv) $p(x) = (x-1)(x+2);\ x = -1, -2$

(v) $p(y) = y^2;\ y = 0$ (vi) $p(x) = ax + b;\ x = -\dfrac{b}{a}$

(vii) $f(x) = 3x^2 - 1;\ x = -\dfrac{1}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}$ (viii) $f(x) = 2x - 1,\ x = \dfrac{1}{2}, \dfrac{-1}{2}$

4. Find the zero of the polynomial in each of the following cases.

(i) $f(x) = x + 2$ (ii) $f(x) = x - 2$ (iii) $f(x) = 2x + 3$

(iv) $f(x) = 2x - 3$ (v) $f(x) = x^2$ (vi) $f(x) = px,\ p \neq 0$

(vii) $f(x) = px + q,\ p \neq 0,\ p\ q$ are real numbers.

5. If 2 is a zero of the polynomial $p(x) = 2x^2 - 3x + 7a$, find the value of a.

6. If 0 and 1 are the zeroes of the polynomial $f(x) = 2x^3 - 3x^2 + ax + b$, find the values of a and b.

2.5 DIVIDING POLYNOMIALS

Observe the following examples

(i) Let us consider two numbers 25 and 3. Divide 25 by 3. We get the quotient 8 and remainder 1. We write

Dividend = (Divisor × Quotient) + Remainder

So, $25 = (8 \times 3) + 1$

Similarly divide 20 by 5, we get $20 = (4 \times 5) + 0$

The remainder here is 0. In this case we say that 5 is a **factor** of 20 or 20 is a **multiple of 5**.

As we divide a number by another non-zero number, we can also divide a polynomial by another polynomial? Let's see.

(ii) Divide the polynomial $3x^3+x^2+x$ by the monomial x.

We have $(3x^3+x^2+x) \div x = \dfrac{3x^3}{x}+\dfrac{x^2}{x}+\dfrac{x}{x}$

$= 3x^2+x+1$

In fact x is a common factor to each term of $3x^3+x^2+x$ So we can write

$3x^3+x^2+x$ as $x(3x^2+x+1)$

What are the factors of $3x^3+x^2+x$?

(iii) Consider another example $(2x^2+x+1) \div x$

Here, $(2x^2+x+1) \div x = \dfrac{2x^2}{x}+\dfrac{x}{x}+\dfrac{1}{x}$

$= 2x+1+\dfrac{1}{x}$

$$\begin{array}{r|l} & \quad\; 2x+1 \\ \hline x & 2x^2+x+1 \\ & -2x^2 \\ \hline & \quad\quad\; x+1 \\ & \quad\quad -x \\ \hline & \quad\quad\quad\; 1 \\ \hline \end{array}$$

Is it a polynomial ?

As one of the term $\dfrac{1}{x}$ has a negative integer exponent (i.e. $\dfrac{1}{x}=x^{-1}$)

$\therefore\ 2x+1+\dfrac{1}{x}$ is not a polynomial.

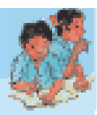
We can however write

$(2x^2+x+1) = [x \times (2x+1)]+1$

By taking out 1 separately the rest of the polynomial can be written as product of two polynomials.

Here we can say $(2x+1)$ is the quotient, x is the divisor and 1 is the remainder. We must keep in mind that since the remainder is not zero, 'x' is not a factor of $2x^2+x+1$.

DO THESE



1. Divide $3y^3+2y^2+y$ by 'y' and write division fact
2. Divide $4p^2+2p+2$ by '$2p$' and write division fact.

Example-6. Divide $3x^2+x-1$ by $x+1$.

Solution : Consider $p(x) = 3x^2+x-1$ and $q(x) = x+1$.

Divide $p(x)$ by $q(x)$. Recall the division process you have learnt in earlier classes.

Step 1: Divide $\dfrac{3x^2}{x}=3x$, it becomes first term in quotient.

Step 2 : Multiply $(x+1)\ 3x = 3x^2 + 3x$

by subtracting $3x^2 + 3x$ from $3x^2 + x$, we get $-2x$

Step 3 : Divide $\dfrac{-2x}{x} = -2$, it becomes the 2nd term in the quotient.

Step 4 : Multiply $(x+1)(-2) = -2x - 2$,

Subtract it from $-2x - 1$, which gives '1'.

Step 5 : We stop here as the remainder is 1, a constant.

(Can you tell why a constant is not divided by a polynomial?)

$$
\begin{array}{r|l}
 & 3x - 2 \\ \hline
x+1 & 3x^2 + x - 1 \\
 & 3x^2 + 3x \\
 & - \quad - \\ \hline
 & \quad -2x - 1 \\
 & \quad -2x - 2 \\
 & \quad + \quad + \\ \hline
 & \qquad +1
\end{array}
$$

This gives us the quotient as $(3x-2)$ and remainder $(+1)$.

Note : The division process is said to be complete if we get the remainder 0 or the degree of the remainder is less than the degree of the divisor.

Now, we can write the division fact as

$3x^2 + x - 1 = (x + 1)(3x - 2) + 1$

i.e. Dividend = (Divisor × Quotient) + Remainder.

Let us see by replacing x by -1 in $p(x)$

$p(x) = 3x^2 + x - 1$

$p(-1) = 3(-1)^2 + (-1) - 1$

$= 3(+1) + (-1) - 1 = 1.$

It is observed that the value of $p(-1)$ is equal the remainder; i.e. is '1'.

So, the remainder obtained on dividing $p(x) = 3x^2 + x - 1$ by $(x+1)$ is same as $p(-1)$ where -1 is the zero of $x+1$. i.e. $x = -1$.

Let us consider some more examples.

Example-7. Divide the polynomial $2x^4 - 4x^3 - 3x - 1$ by $(x-1)$ and verify the remainder with zero of the divisor.

Solution : Let $f(x) = 2x^4 - 4x^3 - 3x - 1$

First see how many times $2x^4$ is of x.

$$\frac{2x^4}{x} = 2x^3$$

Now multiply $(x-1)(2x^3) = 2x^4 - 2x^3$

Then again see the first term of the remainder that is $-2x^3$. Now do the same.

$$
\begin{array}{r|l}
 & 2x^3 - 2x^2 - 2x - 5 \\ \hline
x-1 & 2x^4 - 4x^3 - 3x - 1 \\
 & 2x^4 - 2x^3 \\
 & - \quad + \\ \hline
 & \quad -2x^3 - 3x - 1 \\
 & \quad -2x^3 + 2x^2 \\
 & \quad + \quad - \\ \hline
 & \qquad -2x^2 - 3x - 1 \\
 & \qquad -2x^2 + 2x \\
 & \qquad + \quad - \\ \hline
 & \qquad\quad -5x - 1 \\
 & \qquad\quad -5x + 5 \\
 & \qquad\quad + \quad - \\ \hline
 & \qquad\qquad -6 \\ \hline
\end{array}
$$

Here the quotient is $2x^3 - 2x^2 - 2x - 5$ and the remainder is -6.

Now, the zero of the polynomial $(x - 1)$ is 1.

Put $x = 1$ in $f(x)$, $f(x) = 2x^4 - 4x^3 - 3x - 1$

$$\begin{aligned} f(1) &= 2(1)^4 - 4(1)^3 - 3(1) - 1 \\ &= 2(1) - 4(1) - 3(1) - 1 \\ &= 2 - 4 - 3 - 1 \\ &= -6 \end{aligned}$$

Is the remainder same as the value of the polynomial $f(x)$ at zero of $(x - 1)$?

From the above examples we shall now state the fact in the form of the following theorem. It gives a remainder without actual division of a polynomial by a **linear polynomial in one variable**.

Remainder Theorem :

Let $p(x)$ be any polynomial of degree greater than or equal to one and let 'a' be any real number. If $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.

Let us now look at the proof of this theorem.

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1.

Further suppose that when $p(x)$ is divided by a linear polynomial $g(x) = (x - a)$, the quotient is $q(x)$ and the remainder is $r(x)$. In other words, $p(x)$ and $g(x)$ are two polynomials such that the degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$ then we can find polynomials $q(x)$ and $r(x)$ such that, where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

By division algorithm,

$p(x) = g(x) \,.\, q(x) + r(x)$

$\therefore\ p(x) = (x - a) \,.\, q(x) + r(x)$ $\qquad \because\ g(x) = (x - a)$

Since the degree of $(x - a)$ is 1 and the degree of $r(x)$ is less than the degree of $(x - a)$.

$\therefore$ Degree of $r(x) = 0$, implies $r(x)$ is a constant, say K.

So, for every real value of x, $r(x) = \text{K}$.

Therefore,

$$p(x) = (x - a)\, q(x) + \text{K}$$

If $x = a$, then $p(a) = (a - a)\, q(a) + \text{K}$

$$= 0 + \text{K}$$
$$= \text{K}$$

Hence proved.

Let us use this result in finding remainders when a polynomial is divided by a linear polynomial without actual division.

Example-8. Find the remainder when $x^3 + 1$ divided by $(x + 1)$

Solution : Here $p(x) = x^3 + 1$

The zero of the linear polynomial $x + 1$ is -1 $[x + 1 = 0,\ x = -1]$

So replacing x by -1

$$\begin{aligned} p(-1) &= (-1)^3 + 1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

So, by Remainder Theorem, we know that $(x^3 + 1)$ divided by $(x + 1)$ gives 0 as the remainder.

You can also check this by actually dividing $x^3 + 1$ by $x + 1$.

Can you say $(x + 1)$ is a factor of $(x^3 + 1)$?

Example-9. Check whether $(x - 2)$ is a factor of $x^3 - 2x^2 - 5x + 4$

Solution : Let $p(x) = x^3 - 2x^2 - 5x + 4$

To check whether the linear polynomial $(x - 2)$ is a factor of the given polynomial,

Replace x, by the zero of $(x - 2)$ i.e. $x - 2 = 0 \Rightarrow x = 2$.

$$\begin{aligned} p(2) &= (2)^3 - 2(2)^2 - 5(2) + 4 \\ &= 8 - 2(4) - 10 + 4 \\ &= 8 - 8 - 10 + 4 \\ &= -6. \end{aligned}$$

As the remainder is not equal to zero, the polynomial $(x - 2)$ is not a factor of the given polynomial $x^3 - 2x^2 - 5x + 4$.

Example10. Check whether the polynomial $p(y) = 4y^3 + 4y^2 - y - 1$ is a multiple of $(2y + 1)$.

Solution : As you know, $p(y)$ will be a multiple of $(2y + 1)$ only, if $(2y + 1)$ divides $p(y)$ exactly.

We shall first find the zero of the divisor , $2y + 1$, i.e., $y = \dfrac{-1}{2}$,

Replace y by $\frac{-1}{2}$ in $p(y)$

$$p\left(\frac{-1}{2}\right) = 4\left(\frac{-1}{2}\right)^3 + 4\left(\frac{-1}{2}\right)^2 - \left(\frac{-1}{2}\right) - 1$$

$$= 4\left(\frac{-1}{8}\right) + 4\left(\frac{1}{4}\right) + \frac{1}{2} - 1$$

$$= \frac{-1}{2} + 1 + \frac{1}{2} - 1$$

$$= 0$$

So, $(2y+1)$ is a factor of $p(y)$. That is $p(y)$ is a multiple of $(2y+1)$.

Example-11. If the polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $(x-2)$ leave the same remainder, find the value of a.

Solution : Let $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$

$\because$ $p(x)$ and $q(x)$ when divided by $x-2$ leave same remainder.

$\therefore$ $p(2) = q(2)$

$$a(2)^3 + 3(2)^2 - 13 = 2(2)^3 - 5(2) + a$$

$$8a + 12 - 13 = 16 - 10 + a$$

$$8a - 1 = a + 6$$

$$8a - a = 6 + 1$$

$$7a = 7$$

$$a = 1$$

EXERCISE - 2.3

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by the following Linear polynomials :

   (i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x (iv) $x + \pi$

   (v) $5 + 2x$

2. Find the remainder when $x^3 - px^2 + 6x - p$ is divided by $x - p$.

3. Find the remainder when $2x^2 - 3x + 5$ is divided by $2x - 3$. Does it exactly divide the polynomial ? State reason.

4. Find the remainder when $9x^3 - 3x^2 + x - 5$ is divided by $x - \frac{2}{3}$

5. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by $x - 2$, find the value of a.

6. If the polynomials $x^3 + ax^2 + 5$ and $x^3 - 2x^2 + a$ are divided by $(x + 2)$ leave the same remainder, find the value of a.

7. Find the remainder when $f(x) = x^4 - 3x^2 + 4$ is divided by $g(x) = x - 2$ and verify the result by actual division.

8. Find the remainder when $p(x) = x^3 - 6x^2 + 14x - 3$ is divided by $g(x) = 1 - 2x$ and verify the result by long division.

9. When a polynomial $2x^3 + 3x^2 + ax + b$ is divided by $(x - 2)$ leaves remainder 2, and $(x + 2)$ leaves remainder -2. Find a and b.

2.6 FACTORISING A POLYNOMIAL

As we have already studied that a polynomial $q(x)$ is said to have divided a polynomial $p(x)$ exactly if the remainder is zero. In this case $q(x)$ is a factor of $p(x)$.

For example. When $p(x) = 4x^3 + 4x^2 - x - 1$ is divided by $g(x) = 2x + 1$, if the remainder is zero (verify)

then $4x^3 + 4x^2 - x - 1 = q(x)\,(2x + 1) + 0$

So $p(x) = q(x)\,(2x + 1)$

Therefore $g(x) = 2x + 1$ is a factor of $p(x)$.

With the help of Remainder Theorem can you state a theorem that helps find the factors of a given polynomial ?

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and 'a' is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$ (ii) and its converse "if $(x - a)$ is a factor of a polynomial $p(x)$ then $p(a) = 0$.

Let us see the simple proof of this theorem.

Proof : By Remainder Theorem,

$$p(x) = (x - a)\; q(x) + p(a)$$

(i) Consider proposition (i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x) + 0$.

$$= (x - a)\, q(x)$$

Which shows that $(x-a)$ is a factor of $p(x)$.

Hence proved.

(ii) Consider proposition (ii) Since $(x-a)$ is a factor of $p(x)$, then $p(x) = (x-a)\,q(x)$ for some polynomial $q(x)$

$$\therefore\ p(a) = (a-a)\,q(a) = 0$$

$\therefore$ Hence $p(a) = 0$ when $(x-a)$ is a factor of $p(x)$

Let us consider some examples.

Example-12. Examine whether $x+2$ is a factor of $x^3 + 2x^2 + 3x + 6$

Solution : Let $p(x) = x^3 + 2x^2 + 3x + 6$ and $g(x) = x + 2$

The zero of $g(x)$ is -2

Then
$$\begin{aligned} p(-2) &= (-2)^3 + 2(-2)^2 + 3(-2) + 6 \\ &= -8 + 2(4) - 6 + 6 \\ &= -8 + 8 - 6 + 6 \\ &= 0 \end{aligned}$$

So, by the Factor Theorem, $x+2$ is a factor of $x^3 + 2x^2 + 3x + 6$.

Example-13. Find the value of K, if $2x-3$ is a factor of $2x^3 - 9x^2 + x + \mathrm{K}$.

Solution : $(2x-3)$ is a factor of $p(x) = 2x^3 - 9x^2 + x + \mathrm{K}$,

If $(2x-3) = 0$, $x = \dfrac{3}{2}$

$\therefore$ The zero of $(2x-3)$ is $\dfrac{3}{2}$

If $(2x-3)$ is a factor of $p(x)$ then $p\left(\dfrac{3}{2}\right) = 0$

$p(x) = 2x^3 - 9x^2 + x + \mathrm{K}$,

$$\Rightarrow p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + \mathrm{K} = 0$$

$$\Rightarrow 2\left(\frac{27}{8}\right) - 9\left(\frac{9}{4}\right) + \frac{3}{2} + \mathrm{K} = 0$$

$$\Rightarrow \left(\frac{27}{4} - \frac{81}{4} + \frac{3}{2} + \mathrm{K} = 0\right) \times 4$$

$$27 - 81 + 6 + 4K = 0$$
$$-48 + 4K = 0$$
$$4K = 48$$
So $$K = 12$$

Example-14. Show that $(x - 1)$ is a factor of $x^{10} - 1$ and also of $x^{11} - 1$.

Solution : Let $p(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

To prove $(x - 1)$ is a factor of both $p(x)$ and $g(x)$, it is sufficient to show that $p(1) = 0$ and $g(1) = 0$.

Now

$p(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

$p(1) = (1)^{10} - 1$ and $g(1) = (1)^{11} - 1$

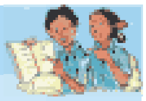
$= 1 - 1$ $\quad$ $= 1 - 1$

$= 0$ $\quad$ $= 0$

Thus by Factor Theorem,

$(x - 1)$ is a factor of both $p(x)$ and $g(x)$.

TRY THESE



Show that $(x - 1)$ is a factor of $x^n - 1$.

We shall now try to factorise quadratic polynomial of the type $ax^2 + bx + c$, (where $a \neq 0$ and a, b, c are constants).

Let its factors be $(px + q)$ and $(rx + s)$.

Then $ax^2 + bx + c = (px + q)\ (rx + s)$

$$= prx^2 + (ps + qr)x + qs$$

By comparing the coefficients of x^2, x and constants, we get that,

$$a = pr$$
$$b = ps + qr$$
$$c = qs$$

This shows that b is the sum of two numbers ps and qr,

whose product is $(ps)\ (qr) = (pr)(qs)$

$$= ac$$

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac.

Example-15. Factories $3x^2 + 11x + 6$

Solution : If we can find two numbers p and q such that $p + q = 11$ and $pq = 3 \times 6 = 18$, then we can get the factors.

So, let us see the pairs of factors of 18.

(1, 18), (2, 9), (3, 6) of these pairs, 2 and 9 will satisfy $p + q = 11$

So $\quad 3x^2 + 11x + 6 = 3x^2 + 2x + 9x + 6$

$$= x(3x + 2) + 3(3x + 2)$$

$$= (3x + 2)(x + 3).$$

DO THESE

Factorise the following

1. $6x^2 + 19x + 15$ $\quad$ 2. $10m^2 - 31m - 132$ $\quad$ 3. $12x^2 + 11x + 2$

Now, consider an example.

Example-16. Verify whether $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is divisible by $x^2 - 3x + 2$ or not ? How can you verify using Factor Theorem ?

Solution : The divisor is not a linear polynomial. It is a quadratic polynomial. You have learned the factorisation of a quadratic polynomial by splitting the middle term as follows.

$$x^2 - 3x + 2 = x^2 - 2x - x + 2$$

$$= x(x - 2) - 1(x - 2)$$

$$= (x - 2)(x - 1).$$

To show $x^2 - 3x + 2$ is a factor of polynomial $2x^4 - 6x^3 + 3x^2 + 3x - 2$, we have to show $(x - 2)$ and $(x - 1)$ are the factors of $2x^4 - 6x^3 + 3x^2 + 3x - 2$.

Let $\quad p(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

then $\quad p(2) = 2(2)^4 - 6(2)^3 + 3(2)^2 + 3(2) - 2$

$$= 2(16) - 6(8) + 3(4) + 6 - 2$$

$$= 32 - 48 + 12 + 6 - 2$$

$$= 50 - 50$$

$$= 0$$

As $p(2) = 0$, $(x - 2)$ is a factor of $p(x)$.

Then $p(1) = 2(1)^4 - 6(1)^3 + 3(1)^2 + 3(1) - 2$

$= 2(1) - 6(1) + 3(1) + 3 - 2$

$= 2 - 6 + 3 + 3 - 2$

$= 8 - 8$

$= 0$

As $p(1) = 0$, $(x - 1)$ is a factor of $p(x)$.

As both $(x - 2)$ and $(x - 1)$ are factors of $p(x)$, the product $x^2 - 3x + 2$ will also be a factor of $p(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$.

Example-17. Factorise $x^3 - 23x^2 + 142x - 120$

Solution : Let $p(x) = x^3 - 23x^2 + 142x - 120$

By trial, we find that $p(1) = 0$. (verify)

So $(x - 1)$ is a factor of $p(x)$

When we divide $p(x)$ by $(x - 1)$, we get $x^2 - 22x + 120$.

Another way of doing this is,

$$x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$$

$$= x^2(x - 1) - 22x(x - 1) + 120(x - 1) \quad \text{(why?)}$$

$$= (x - 1)(x^2 - 22x + 120)$$

Now $x^2 - 22x + 120$ is a quadratic expression that can be factorised by splitting the middle term. We have

$$x^2 - 22x + 120 = x^2 - 12x - 10x + 120$$

$$= x(x - 12) - 10(x - 12)$$

$$= (x - 12)(x - 10)$$

So, $x^3 - 23x^2 + 142x - 120 = (x - 1)(x - 10)(x - 12)$.

EXERCISE - 2.4

1. Determine which of the following polynomials has $(x+1)$ as a factor.

 (i) $x^3 - x^2 - x + 1$ (ii) $x^4 - x^3 + x^2 - x + 1$

 (iii) $x^4 + 2x^3 + 2x^2 + x + 1$ (iv) $x^3 - x^2 - (3 - \sqrt{3})\,x + \sqrt{3}$

2. Use the Factor Theorem to determine whether g(x) is factor of $f(x)$ in each of the following cases :

 (i) $f(x) = 5x^3 + x^2 - 5x - 1,\ g(x) = x + 1$

 (ii) $f(x) = x^3 + 3x^2 + 3x + 1,\ g(x) = x + 1$

 (iii) $f(x) = x^3 - 4x^2 + x + 6,\ g(x) = x - 2$

 (iv) $f(x) = 3x^3 + x^2 - 20x + 12,\ g(x) = 3x - 2$

 (v) $f(x) = 4x^3 + 20x^2 + 33x + 18,\ g(x) = 2x + 3$

3. Show that $(x - 2)$, $(x + 3)$ and $(x - 4)$ are factors of $x^3 - 3x^2 - 10x + 24$.

4. Show that $(x + 4)$, $(x - 3)$ and $(x - 7)$ are factors of $x^3 - 6x^2 - 19x + 84$.

5. If both $(x - 2)$ and $\left(x - \frac{1}{2}\right)$ are factors of $px^2 + 5x + r$, show that $p = r$.

6. If $(x^2 - 1)$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$

7. Factorize (i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

 (iii) $x^3 + 13x^2 + 32x + 20$ (iv) $y^3 + y^2 - y - 1$

8. If $ax^2 + bx + c$ and $bx^2 + ax + c$ have a common factor $x + 1$ then show that $c = 0$ and $a = b$.

9. If $x^2 - x - 6$ and $x^2 + 3x - 18$ have a common factor $(x - a)$ then find the value of a.

10. If $(y - 3)$ is a factor of $y^3 - 2y^2 - 9y - 18$ then find the other two factors.

2.6 ALGEBRAIC IDENTITIES

Recall that an algebraic Identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes

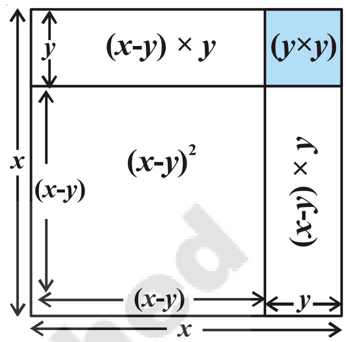
Identity I : $(x + y)^2 \equiv x^2 + 2xy + y^2$

Identity II : $(x - y)^2 \equiv x^2 - 2xy + y^2$

Identity III : $(x+y)(x-y) \equiv x^2 - y^2$

Identity IV : $(x+a)(x+b) \equiv x^2 + (a+b)x + ab.$

Geometrical Proof :



For Identity $(x-y)^2$

Step-I Make a square of side x.

Step-II Subtract length y from x.

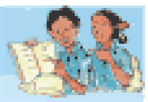
Step-III Calculate for $(x-y)^2$

$= x^2 - [(x-y)\,y + (x-y)\,y + y^2]$

$= x^2 - xy + y^2 - xy + y^2 - y^2$

$= x^2 - 2xy + y^2$

TRY THIS

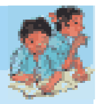


Try to draw the geometrical figures for other identities.

(i) $(x+y)^2 \equiv x^2 + 2xy + y^2$ (ii) $(x+y)(x-y) \equiv x^2 - y^2$

(iii) $(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

DO THESE



Find the following product using appropriate identities

(i) $(x+5)(x+5)$ (ii) $(p-3)(p+3)$ (iii) $(y-1)(y-1)$

(iv) $(t+2)(t+4)$ (v) 102×98

Identities are useful in factorisation of algebraic expressions. Let us see some examples.

Example-18. Factorise

(i) $x^2 + 5x + 4$ (ii) $9x^2 - 25$

(iii) $25a^2 + 40ab + 16b^2$ (iv) $49x^2 - 112xy + 64y^2$

Solution :

(i) Here $x^2 + 5x + 4 = x^2 + (4+1)x + (4)(1)$

Comparing with Identity $(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

we get $(x+4)(x+1)$.

(ii) $9x^2 - 25 = (3x)^2 - (5)^2$

Now comparing it with Identity III, $x^2 - y^2 \equiv (x+y)(x-y)$, we get

$\therefore\ 9x^2 - 25 = (3x+5)(3x-5)$.

(iii) Here you can see that

$25a^2 + 40ab + 16b^2 = (5a)^2 + 2(5a)(4b) + (4b)^2$

Comparing this expression with $x^2 + 2xy + y^2$,

we observe that $x = 5a$ and $y = 4b$

Using Identity I, $\mathbf{(x + y)^2 \equiv x^2 + 2xy + y^2}$

we get $25a^2 + 40ab + 16b^2 = (5a + 4b)^2$

$= (5a + 4b)(5a + 4b).$

(iv) Here $49x^2 - 112xy + 64y^2$, we see that

$49x^2 = (7x)^2$, $64y^2 = (8y)^2$ and

$112\,xy = 2(7x)(8y)$

Thus comparing with Identity II,

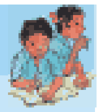
$\mathbf{(x - y)^2 \equiv x^2 - 2xy + y^2}$,

we get, $49x^2 - 112xy + 64y^2 = (7x)^2 - 2(7x)(8y) + (8y)^2$

$= (7x - 8y)^2$

$= (7x - 8y)(7x - 8y).$

DO THESE



Factorise the following using appropriate identities

(i) $49a^2 + 70ab + 25b^2$ (ii) $\frac{9}{16}x^2 - \frac{y^2}{9}$

(iii) $t^2 - 2t + 1$ (iv) $x^2 + 3x + 2$

So far, all our identities involved products of binomials. Let us now extend the identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$.

Let $x + y = t$, then $(x + y + z)^2 = (t + z)^2$

$= t^2 + 2tz + z^2$ (using Identity I)

$= (x + y)^2 + 2(x + y)z + z^2$ (substituting the value of 't')

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$

By rearranging the terms, we get $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Alternate Method :

You can also compute $(x+y+z)^2$ by regrouping the terms

$$[(x+y)+z]^2 = (x+y)^2 + 2(x+y)(z) + (z)^2$$
$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \qquad \text{[From identity (1)]}$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

In what other ways you can regroup the terms to find the expansion ? Will you get the same result ?

So, we get the following Identity

Identity V : $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$

Example-19. Expand $(2a+3b+5)^2$ using identity.

Solution : Comparing the given expression with $(x+y+z)^2$,

we find that $x = 2a$, $y = 3b$ and $z = 5$

Therefore, using Identity V, we have

$$(2a+3b+5)^2 = (2a)^2 + (3b)^2 + (5)^2 + 2(2a)(3b) + 2(3b)(5) + 2(5)(2a)$$
$$= 4a^2 + 9b^2 + 25 + 12ab + 30b + 20a.$$

Example-20. Find the product of $(5x-y+z)(5x-y+z)$

Solution : Here $(5x-y+z)(5x-y+z) = (5x-y+z)^2$

$$= [5x + (-y) + z]^2$$

Therefore using the Identity V, $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$, we get

$$(5x+(-y)+z)^2 = (5x)^2 + (-y)^2 + (z)^2 + 2(5x)(-y) + 2(-y)(z) + 2(z)(5x)$$
$$= 25x^2 + y^2 + z^2 - 10xy - 2yz + 10zx.$$

Example-21. Factorise $4x^2 + 9y^2 + 25z^2 - 12xy - 30yz + 20zx$

Solution : We have

$$4x^2 + 9y^2 + 25z^2 - 12xy - 30yz + 20zx$$
$$= [(2x)^2 + (-3y)^2 + (5z)^2 + 2(2x)(-3y) + 2(-3y)(5z) + 2(5z)(2x)]$$

Comparing with the identity V,

$(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$, we get

$$= (2x-3y+5z)^2$$

$$= (2x-3y+5z)\,(2x-3y+5z).$$

DO THESE

(i) Write $(p+2q+r)^2$ in expanded form.

(ii) Expand $(4x-2y-3z)^2$ using identity

(iii) Factorise $4a^2+b^2+c^2-4ab+2bc-4ca$ using identity.

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to find $(x+y)^3$.

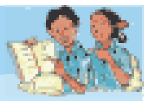
We have

$$\begin{aligned}(x+y)^3 &= (x+y)^2\,(x+y)\\ &= (x^2+2xy+y^2)\,(x+y)\\ &= x(x^2+2xy+y^2)+y(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3\\ &= x^3+3xy\,(x+y)+y^3\\ &= x^3+y^3+3xy(x+y).\end{aligned}$$

So, we get the following identity.

Identity VI : $(x+y)^3 \equiv x^3+y^3+3xy\,(x+y)$.

TRY THESE



How can you find $(x-y)^3$ without actual multiplication ?
Verify with actual multiplication.

You get the next identity as

Identity VII : $(x-y)^3 \equiv x^3-y^3-3xy\,(x-y)$.

$$\equiv x^3-3x^2y+3xy^2-y^3$$

Let us see some examples where these identities are being used.

Example-22. Write the following cubes in the expanded form

(i) $(2a+3b)^3$ (ii) $(2p-5)^3$

Solution : (i) Comparing the given expression with $(x+y)^3$, we find that $x=2a$ and $y=3b$

So, using Identity VI, we have

$$(2a+3b)^3 = (2a)^3 + (3b)^3 + 3(2a)(3b)\,(2a+3b)$$
$$= 8a^3 + 27b^3 + 18ab\,(2a+3b)$$
$$= 8a^3 + 27b^3 + 36a^2b + 54\,ab^2$$
$$= 8a^3 + 36a^2b + 54ab^2 + 27b^3.$$

(ii) Comparing the given expression with $(x-y)^3$, we find that $x=2p$ and $y=5$

So, using Identity VII , we have

$$(2p-5)^3 = (2p)^3 - (5)^3 - 3(2p)(5)\,(2p-5)$$
$$= 8p^3 - 125 - 30p\,(2p-5)$$
$$= 8p^3 - 125 - 60p^2 + 150p$$
$$= 8p^3 - 60p^2 + 150p - 125.$$

Example-23. Evaluate each of the following using suitable identities

(i) $(103)^3$ (ii) $(99)^3$

Solution : (i) We have

$(103)^3 = (100+3)^3$

Comparing with $(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$ we get

$$= (100)^3 + (3)^3 + 3(100)\,(3)\,(100+3)$$
$$= 1000000 + 27 + 900(103)$$
$$= 1000000 + 27 + 92700$$
$$= 1092727.$$

(ii) We have $(99)^3 = (100-1)^3$

Comparing with $(x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$ we get

$$= (100)^3 - (1)^3 - 3(100)(1)\,(100-1)$$
$$= 1000000 - 1 - 300\,(99)$$

$= 1000000 - 1 - 29700$

$= 970299.$

Example-24. Factorise $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

Solution : The given expression can be written as

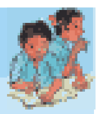
$$8x^3 + 36x^2y + 54xy^2 + 27y^3 = (2x)^3 + 3(2x)^2\,(3y) + 3(2x)\,(3y)^2 + (3y)^3$$

Comparing with Identity VI, $(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$,

we get $= (2x + 3y)^3$

$= (2x + 3y)\,(2x + 3y)\,(2x + 3y).$

DO THESE



1. Expand $(x+1)^3$ using an identity
2. Compute $(3m-2n)^3$.
3. Factorise $a^3 - 3a^2b + 3ab^2 - b^3$.

Now consider $(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

on expanding, we get the product as

$$= x(x^2 + y^2 + z^2 - xy - yz - zx) + y(x^2 + y^2 + z^2 - xy - yz - zx) + z(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= x^3 + \cancel{xy^2} + \cancel{xz^2} - \cancel{x^2y} - xyz - \cancel{x^2z} + \cancel{x^2y} + y^3 + \cancel{yz^2} - \cancel{xy^2} - \cancel{y^2z} - xyz + \cancel{x^2z} + \cancel{y^2z} + z^3 - xyz - \cancel{yz^2} - \cancel{xz^2}$$

$$= x^3 + y^3 + z^3 - 3xyz \text{ (on simplification)}$$

Thus

Identity VIII : $(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - xz) \equiv x^3 + y^3 + z^3 - 3xyz$

Example-25. Find the product

$$(2a + b + c)\,(4a^2 + b^2 + c^2 - 2ab - bc - 2ca)$$

Solution : Here the product that can be written as

$$= (2a+b+c)\,[(2a)^2+b^2+c^2-(2a)(b)-(b)(c)-(c)\,(2a)]$$

Comparing with Identity VIII,

$$(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx) \equiv x^3+y^3+z^3-3xyz$$

$$= (2a)^3+(b)^3+(c)^3-3(2a)\,(b)\,(c)$$

$$= 8a^3+b^3+c^3-6abc$$

Example-26. Factorise $a^3 - 8b^3 - 64c^3 - 24abc$

Solution : Here the given expression can be written as

$$a^3 - 8b^3 - 64c^3 - 24abc = (a)^3 + (-2b)^3 + (-4c)^3 - 3(a)(-2b)(-4c)$$

Comparing with the identity VIII,

$$x^3+y^3+z^3 - 3xyz \equiv (x+y+z)\,(x^2+y^2+z^2 - xy - yz - zx)$$

we get factors as

$$= (a-2b-4c)\,[(a)^2+(-2b)^2+(-4c)^2 - (a)\,(-2b) - (-2b)\,(-4c) - (-4c)\,(a)]$$

$$= (a-2b-4c)\,(a^2+4b^2+16c^2+2ab-8bc+4ca).$$

DO THESE

1. Find the product $(a-b-c)\,(a^2+b^2+c^2-ab+bc-ca)$ without actual multiplication.
2. Factorise $27a^3+b^3+8c^3-18abc$ using identity.

Example-27. Give possible values for length and breadth of the rectangle whose area is $2x^2+9x-5$.

Solution : Let l, b be length and breadth of a rectangle

Area of rectangle $= 2x^2+9x-5$

$$lb = 2x^2+9x-5$$

$$= 2x^2+10x-x-5$$

$$= 2x(x+5)-1(x+5)$$

$$= (x+5)\,(2x-1)$$

$\therefore$ length $= (x+5)$

breadth $= (2x-1)$

Let $x = 1,\ l = 6,\ b = 1$

$x = 2,\ l = 7,\ b = 3$

$x = 3,\ l = 8,\ b = 5$

....

....

Can you find more values ?

EXERCISE - 2.5

1. Use suitable identities to find the following products

   (i) $(x+5)(x+2)$ (ii) $(x-5)(x-5)$ (iii) $(3x+2)(3x-2)$

   (iv) $\left(x^2+\dfrac{1}{x^2}\right)\left(x^2-\dfrac{1}{x^2}\right)$ (v) $(1+x)(1+x)$

2. Evaluate the following products without actual multiplication.

   (i) 101×99 (ii) 999×999 (iii) $50\dfrac{1}{2}\times 49\dfrac{1}{2}$

   (iv) 501×501 (v) 30.5×29.5

3. Factorise the following using appropriate identities.

   (i) $16x^2 + 24xy + 9y^2$ (ii) $4y^2 - 4y + 1$

   (iii) $4x^2 - \dfrac{y^2}{25}$ (iv) $18a^2 - 50$

   (v) $x^2 + 5x + 6$ (vi) $3p^2 - 24p + 36$

4. Expand each of the following, using suitable identities

   (i) $(x+2y+4z)^2$ (ii) $(2a-3b)^3$ (iii) $(-2a+5b-3c)^2$

   (iv) $\left(\dfrac{a}{4}-\dfrac{b}{2}+1\right)^2$ (v) $(p+1)^3$ (vi) $\left(x-\dfrac{2}{3}y\right)^3$

5. Factorise

   (i) $25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$

   (ii) $9a^2 + 4b^2 + 16c^2 + 12ab - 16bc - 24ca$

6. If $a+b+c=9$ and $ab+bc+ca=26$, find $a^2+b^2+c^2$.

7. Evaluate the following using suitable identites.

(i) $(99)^3$ (ii) $(102)^3$ (iii) $(998)^3$ (iv) $(1001)^3$

8. Factorise each of the following

(i) $8a^3+b^3+12a^2b^2+6ab^2$ (ii) $8a^3-b^3-12a^2b^2+6ab^2$

(iii) $1-64a^3-12a+48a^2$ (iv) $8p^3-\frac{12}{5}p^2+\frac{6}{25}p-\frac{1}{125}$

9. Verify (i) $x^3+y^3=(x+y)(x^2-xy+y^2)$ (ii) $x^3-y^3=(x-y)(x^2+xy+y^2)$

using some non-zero positive integers and check by actual multiplication. Can you call these as identites ?

10. Factorise (i) $27a^3+64b^3$ (ii) $49y^3-1000$ using the above results.

11. Factorise $27x^3+y^3+z^3-9xyz$ using identity.

12. Verify that $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2]$

13. If $x+y+z=0$, show that $x^3+y^3+z^3=3xyz$.

14. Without actual calculating the cubes, find the value of each of the following

(i) $(-10)^3+(7)^3+(3)^3$ (ii) $(28)^3+(-15)^3+(-13)^3$

(iii) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3-\left(\frac{5}{6}\right)^3$ (iv) $(0.2)^3-(0.3)^3+(0.1)^3$

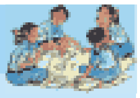
15. Give possible expressions for the length and breadth of the rectangle whose area is given by (i) $4a^2+4a-3$ (ii) $25a^2-35a+12$

16. What are the possible polynomial expressions for the dimensions of the cuboids whose volumes are given below?

(i) $3x^2-12x$ (ii) $12y^2+8y-20$.

17. Show that if $2(a^2+b^2)=(a+b)^2$, then $a=b$

WHAT WE HAVE DISCUSSED



In this chapter, you have studied the following points.

1. A polynomial $p(x)$ in one variable x is an algebraic expression in x of the form $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, a_2, \ldots. a_n$ are respectively the coefficients of $x^0, x^1, x^2, \ldots. x^n$ and n is called the **degree** of the polynomial if $a_n \neq 0$. Each a_nx^n ; $a_{n-1}x^{n-1}$; .... a_0, is called **a term** of the polynomial $p(x)$.
2. Polynomials are classified as monomial, binomial, trinomial etc. according to the number of terms in it.
3. Polynomials are also named as linear polynomial, quadratic polynomial, cubic polynomial etc. according to the degree of the polynomial.
4. A real number 'a' is a zero of a polynomial $p(x)$ if $p(a) = 0$. In this case, 'a' is also called a root of the polynomial equation $p(x) = 0$.
5. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero.
6. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.
7. Factor Theorem : If $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$. Also if $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$.
8. Some Algebraic Identities are:

   (i) $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

   (ii) $(x + y)^3 \equiv x^3 + y^3 + 3xy(x + y)$

   (iii) $(x - y)^3 \equiv x^3 - y^3 - 3xy(x - y)$

   (iv) $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ also

   (v) $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$

   (vi) $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$

Brain teaser

If $\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = \sqrt{x\sqrt{x\sqrt{x\ldots}}}$ then what is the value of x.